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Die Titelseite zeigt die epidemiologische Kurve von Covid-19 in Österreich im ersten Jahr der Pandemie. Die Daten stammen vom Dashboard der AGES. In diesem Jahr wurde “flatten the curve” zu einem viel zitierten Schlagwort, die effektive Reproduktionszahl, die 7-Tage-Inzidenz und andere Kennwerte sind einer informierten Öffentlichkeit nun bestens vertraut, exponentielles Wachstum ist für alle greifbarer geworden. In der Krise haben Mathematikerinnen und Mathematiker wichtige Sichtweisen beigesteuert und gemeinsam mit anderen Expertinnen und Experten somit einen wesentlichen Input für die Entscheidungen der Politik geliefert. Die Krise hat – so wie in allen Bereichen des Lebens – auch in der mathematischen scientific community für etliche Änderungen gesorgt. Videokonferenzen haben unseren Austausch geprägt. Offenbar geht es doch auch ohne Tafel, wenngleich die Diskussion und die Vermittlung von Ideen dabei gelitten hat! Es besteht die berechtigte Hoffnung, dass die gesundheitliche Gefahr mit der Impfung deutlich reduziert wird, sodass die sehr herausfordernden und schwierigen Pandemie-Monate dann letztlich hinter uns liegen werden.

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Quantum information theory and free semialgebraic geometry: one wonderland through two looking glasses

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We illustrate how quantum information theory and free (i.e. noncommutative) semialgebraic geometry often study similar objects from different perspectives. We give examples in the context of positivity and separability, quantum magic squares, quantum correlations in non-local games, and positivity in tensor networks, and we show the benefits of combining the two perspectives. This paper is an invitation to consider the intersection of the two fields, and should be accessible for researchers from either field.

Live free or die.
Motto of New Hampshire

1 Introduction

The ties between physics, computer science and mathematics are historically strong and multidimensional. It has often happened that mathematical inventions which were mere products of imagination (and thus thought to be useless for applications) have later played a crucial role in physics or computer science. A superb example is that of imaginary numbers and their use in complex Hilbert spaces in quantum mechanics.[1] Other examples include number theory and its

[1] Who would have thought that the square root of -1 would have any physical relevance? See [48].

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use in cryptography, or Riemannian geometry and its role in General Relativity. It is also true that physicists tend to be only aware of the mathematical tools useful for them — so there are many branches of mathematics which have not found an outlet in physics.

The relevance of a statement depends on the glass through which we look at it. There are statements which are mathematically unimpressive but physically very impressive. A good example is entanglement. Mathematically, the statement that the positivity cone of the tensor product space is larger than the tensor product of the local cones is interesting, but not particularly wild or surprising. Yet, the physical existence of entangled particles is, from our perspective, truly remarkable. In other words, while the mathematics is easy to understand, the physics is mind-blowing. This is particularly true regarding Bell's Theorem: while it is mathematically not specially deep, we regard the experimental violation of Bell inequalities [29, 33, 50] as very deep indeed. Another example is the no-cloning theorem — it is mathematically trivial, yet it has very far-reaching physical consequences. On the other hand, there are many mathematically impressive statements which are — so far — physically irrelevant. Finally, there are statements which can be both mathematically deep and central for quantum information theory, such as Stinespring's Dilation Theorem.

The goal of this paper is to illustrate how two relatively new disciplines in physics and mathematics — quantum information theory and free semialgebraic geometry — have a lot in common. 'Free' means noncommutative, because it is *free of the commutation relation*. So free semialgebraic geometry studies noncommutative versions of semialgebraic sets. On the other hand, quantum information theory is (mathematically) a noncommutative generalisation of classical information theory. So, intuitively, 'free' is naturally linked to 'quantum'. Moreover, in both fields, positivity plays a very important role. Semialgebraic geometry examines questions arising from nonnegativity, like polynomial inequalities. In quantum information theory, quantum states are represented by positive semidefinite matrices. Positivity also gives rise to convexity, which is central in both fields, as we will see.

So the two disciplines often study the same mathematical objects from different perspectives. As a consequence, they often ask different questions. For example, in quantum information theory, given an element of a tensor product space, one wants to know whether it is positive semidefinite, and how this can be efficiently represented and manipulated. In free semialgebraic geometry, the attention is focused on the geometry of the set of all such elements (see Table 1).

We believe that much is to be learnt by bridging the gap among the two communities — in knowledge, notation and perspective. In this paper we hope to illustrate this point. This paper is thus meant to be accessible for physicists and mathematicians.

Quantum information theory *Emphasis on the element*	Free semialgebraic geometry *Emphasis on the set*
Given $\rho = \sum_\alpha A_\alpha \otimes B_\alpha$, is it positive semidefinite (psd)?	Given $\{A_\alpha\}$, characterise the set of $\{B_\alpha\}$ such that $\sum_\alpha A_\alpha \otimes B_\alpha$ is psd.
Block positive matrices / Separable psd matrices.	Largest / smallest operator system over the psd cone.
Every POVM can be dilated to a PVM.	The free convex hull of the set of PVMs is the set of POVMs.
Can a correlation matrix p be realised by a quantum strategy?	Is p in the free convex hull of the free independence model?

Table 1: Examples of the different approaches of quantum information theory and free semialgebraic geometry in studying essentially the same mathematical objects. The various notions will be explained throughout the paper.

Obviously we are not the first or the only ones to notice these similarities. In this paper, however, we will mainly concentrate on what we have learnt from collaborating in recent years, and we will review a few other works. The selection of other works is not comprehensive and reflects our partial knowledge. We also remark that quantum information theory and free semialgebraic geometry are not the only ones studying positivity in tensor product spaces. *Compositional distributional semantics*, for example, represents the meaning of words by positive semidefinite matrices, and the composition of meanings is thus given by positivity preserving maps — see e.g. [18, 11].

This article is organised as follows. We will first explain basic concepts in quantum information theory and free semialgebraic geometry (Section 2) — the reader familiar with them can skip the corresponding section. Then we will explain how they are related (Section 3), and we will end with some closing words (Section 4).

2 Some basic concepts

Here we present some basic concepts in quantum information theory (Section 2.1) and free semialgebraic geometry (Section 2.2).

Throughout the paper we denote the set of $r \times s$ complex matrices by $\mathrm{Mat}_{r,s}$, the set of $r \times r$ complex matrices by Mat_r, and we use the identification of $\mathrm{Mat}_r \otimes \mathrm{Mat}_s$ with Mat_{rs}. We will also often use the real subspace of Mat_r containing the Hermitian elements, called Her_r, and use that $\mathrm{Her}_r \otimes \mathrm{Her}_s$ is identified with Her_{rs}. The d-fold cartesian product of Her_r is denoted Her_r^d.

2.1 Basic concepts from quantum information theory

Here we briefly introduce some concepts from quantum information theory. We focus on finite-dimensional quantum systems of which we do not assume to have perfect knowledge (in the language of quantum information, these are called *mixed states*). See, e.g. [53, 43], for a more general overview.

The *state* of a quantum system is modelled by a *normalized positive semidefinite matrix*, i.e. a

$$\rho \in \mathrm{Mat}_d \text{ with } \rho \succcurlyeq 0 \text{ and } \mathrm{tr}(\rho) = 1,$$

where $\succcurlyeq 0$ denotes positive semidefinite (psd), i.e. Hermitian with nonnegative eigenvalues, and the trace, tr, is the sum of the diagonal elements. We reserve the symbol $\geqslant 0$ for nonnegative numbers. A *measurement* on the system is modelled by a *positive operator valued measure* (POVM), i.e. a set of psd matrices τ_i that sum to the identity:

$$\tau_1, \ldots, \tau_n \in \mathrm{Mat}_d \text{ with all } \tau_i \succcurlyeq 0 \text{ and } \sum_i \tau_i = I_d.$$

The probability to obtain outcome i on state ρ is given by

$$\mathrm{tr}(\rho\tau_i). \tag{1}$$

Note that these probabilities sum to 1 because of the normalisation condition on the τ_i's and ρ.

When the system is composed of several subsystems, the global state space is modelled as a tensor product of the local spaces,

$$\mathrm{Mat}_d = \mathrm{Mat}_{d_1} \otimes \cdots \otimes \mathrm{Mat}_{d_n}, \tag{2}$$

where $d = d_1 \cdots d_n$.

A state ρ is called *separable* (w.r.t. a given tensor product structure) if it can be written as

$$\rho = \sum_{i=1}^{r} \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(n)} \quad \text{with all } \rho_i^{(j)} \succcurlyeq 0.$$

This is obviously a stronger requirement than ρ being psd — not every ρ is separable. Separable states are not too interesting from a quantum information perspective: *not* separable states are called *entangled*, and entanglement is necessary for many quantum information tasks.

A *quantum channel* is the most general transformation on quantum states. Mathematically, it is modelled by a linear trace-preserving map

$$T\colon \mathrm{Mat}_d \to \mathrm{Mat}_s$$

that is *completely positive*. Complete positivity means that the maps

$$\mathrm{id}_n \otimes T\colon \mathrm{Mat}_{nd} \to \mathrm{Mat}_{ns}$$

are positive (i.e. map psd matrices to psd matrices) for all n, where id_n is the identity map on Mat_n.
Any linear map $T\colon \mathrm{Mat}_d \to \mathrm{Mat}_s$ is uniquely determined by its *Choi matrix*

$$C_T := \sum_{i,j=1}^{d} E_{ij} \otimes T\left(E_{ij}\right) \in \mathrm{Mat}_{ds},$$

where E_{ij} is the matrix with a 1 in the (i,j)-position and 0 elsewhere.[2] It is a basic fact that T is completely positive if and only if C_T is psd (see for example [46, 54]). Moreover, a completely positive map T is *entanglement-breaking* [34] if and only if C_T is a separable matrix, and T is a positive map if and only if C_T is *block positive*, i.e.

$$\mathrm{tr}((\sigma \otimes \tau) C_T) \geqslant 0 \quad \text{for all } \sigma \succcurlyeq 0, \tau \succcurlyeq 0.$$

Note that this is weaker than C_T being psd, in which case $\mathrm{tr}(\chi C_T) \geqslant 0$ for all $\chi \succcurlyeq 0$ (see Table 2). We also remark that this link between positivity notions of linear maps and their Choi matrices does not involve the normalisation conditions on the maps (e.g. preserving the trace) or the matrices (e.g. having a given trace).

Linear map $T : \mathrm{Mat}_d \to \mathrm{Mat}_s$	Element in tensor product space $\rho \in \mathrm{Mat}_d \otimes \mathrm{Mat}_s$
Entanglement-breaking map	Separable matrix
Completely positive map	Positive semidefinite matrix
Positive map	Block positive matrix

Table 2: Correspondence between notions of positivity for linear maps and their Choi matrices. Entanglement-breaking maps are a subset of completely positive maps, which are a subset of positive maps. The same is true for the right column, of course.

[2] If this is expressed in the so-called computational basis (which is one specific orthonormal basis), this is written $E_{ij} = |i\rangle\langle j|$ in quantum information.

2.2 Basic concepts from (free) semialgebraic geometry

We now introduce some basic concepts from free (i.e. noncommutative) semialgebraic geometry. For a slightly more detailed introduction, see [41] and references therein.

Our setup starts by considering a $\mathbb{C}$-vector space V with an involution $*$. The two relevant examples are, first, the case where V is the space of matrices and * is the transposition with complex conjugation — denoted † in quantum information —, and, second, $\mathbb{C}^d$ with entrywise complex conjugation.

The fixed points of the involution are called self-adjoint, or Hermitian, elements. We denote the set of Hermitian elements of V by V_{her}. This is an $\mathbb{R}$-subspace of V, in which the *real* things happen.[3]

In the *free* setup, we do not only consider V but also higher levels thereof. Namely, for any $s \in \mathbb{N}$, we consider the space of $s \times s$-matrices with entries over V,

$$\mathrm{Mat}_s(V) = V \otimes \mathrm{Mat}_s.$$

Recall that Mat_s refers to $s \times s$-matrices with entries over $\mathbb{C}$. $\mathrm{Mat}_s(V)$ is a $\mathbb{C}$-vector space with a 'natural' involution, consisting of transposing and applying $*$ entrywise. This thus promotes V and $*$ to an entire hierarchy of levels, namely $\mathrm{Mat}_s(V)$ for all $s \in \mathbb{N}$ with the just described involution.

We are now ready to define the most general notion of a *free real set*. This is nothing but a collection

$$\mathcal{C} = (C_s)_{s \in \mathbb{N}}$$

where each $C_s \subseteq \mathrm{Mat}_s(V)_{\text{her}} =: \mathrm{Her}_s(V)$. We call C_s the *set at level s*.

To make things more interesting, one often imposes conditions that connect the levels. One important example is *free convexity*, which is defined as follows. For any

$$\tau_i \in C_{t_i} \quad \text{with} \quad i = 1, \ldots, n,$$

and

$$v_i \in \mathrm{Mat}_{t_i,s} \quad \text{with} \quad \sum_{i=1}^{n} v_i^* v_i = I_s, \tag{3}$$

it holds that

$$\sum_{i=1}^{n} v_i^* \tau_i v_i \in C_s. \tag{4}$$

Note that in (4) matrices over the complex numbers (namely v_i) are multiplied with matrices over V (namely τ_i). This is defined as matrix multiplication in the

[3] Because it is where positivity and other interesting phenomena happen.

usual way for $v_i^* \tau_i v_i$, and using that elements of V can be multiplied with complex numbers and added. For example, for $n = 1$, $t = 2$, $s = 1$, $\tau = (\mu_{i,j})$ with $\mu_{i,j} \in V$ for $i = 1, 2$, and $v = (\lambda_1, \lambda_2)^t$ with $\lambda_i \in \mathbb{C}$, we have

$$v^* \tau v = \sum_{i,j} \bar{\lambda}_i \lambda_j \mu_{i,j}.$$

Note that if free convexity holds, then every C_s is a convex set in the real vector space $\mathrm{Her}_s(V)$. But free convexity is generally a stronger condition than 'classical' convexity, as we will see.

In addition, a conic version of free convexity is obtained when giving up the normalization condition on the v_i, i.e. the right hand side of Eq. (3). In this case, $\mathcal{C}$ is called an *abstract operator system* (usually with the additional assumption that every C_s is a proper convex cone).

Now, *free semialgebraic sets* are free sets arising from polynomial inequalities. This will be particularly important for the connection we hope to illustrate in this paper. In order to define these, take $V = \mathbb{C}^d$ with the involution provided by entrywise conjugation, so that $V_{\mathrm{her}} = \mathbb{R}^d$. Let $z_1, \ldots, z_d$ denote free variables, that is, noncommuting variables. We can imagine each z_i to represent a matrix of *arbitrary size* — later we will substitute z_i by a matrix of a given size, and this size will correspond to the level of the free semialgebraic set.

Now let ω be a finite *word* in the letters $z_1, \ldots, z_d$, that is, an ordered tuple of these letters. For example, ω could be $z_1 z_1 z_4$ or $z_4 z_5 z_4$. In addition, let $\sigma_\omega \in \mathrm{Mat}_m$ be a matrix (of some fixed size m) that specifies the coefficients of word ω; this is called the coefficient matrix. A *matrix polynomial* in the free variables $z_1, \ldots, z_d$ is an expression

$$p = \sum_\omega \sigma_\omega \otimes \omega,$$

where the sum is over all finite words ω, and where only finitely many coefficient matrices σ_ω are nonzero.

We denote the reverse of word ω by ω^*. For example, if $\omega = z_1 z_2 z_3$ then $\omega^* = z_3 z_2 z_1$. In addition, $(\sigma_\omega)^*$ is obtained by transposition and complex conjugation of σ_ω. If the coefficient matrices fulfill

$$(\sigma_\omega)^* = \sigma_{\omega^*}, \tag{5}$$

then for any tuple of Hermitian matrices $(\tau_1, \ldots, \tau_d) \in \mathrm{Her}_s^d$ we have that

$$p(\tau_1, \ldots, \tau_d) = \sum_\omega \sigma_\omega \otimes \omega(\tau_1, \ldots, \tau_d) \in \mathrm{Her}_{ms}.$$

That is, p evaluated at the Hermitian matrices $\tau_1, \ldots, \tau_d$ is a Hermitian matrix itself.

So, for a given matrix polynomial p satisfying condition (5), we define the free semialgebraic set at level s as the set of Hermitian matrices of size s such that p evaluated at them is psd:

$$C_s(p) := \left\{ (\tau_1, \ldots, \tau_d) \in \mathrm{Her}_s^d \mid p(\tau_1, \ldots, \tau_d) \succcurlyeq 0 \right\}.$$

Finally we define the free semialgebraic set as the collection of all such levels:

$$\mathcal{C}(p) := (C_s(p))_{s \in \mathbb{N}}.$$

For example, let E_{ii} denote the matrix with a 1 in entry (i,i) and 0 elsewhere. Then the matrix polynomial

$$p = \sum_{i=1}^{d} E_{ii} \otimes z_i$$

defines the following free semialgebraic set at level s

$$C_s(p) = \left\{ (\tau_1, \ldots, \tau_d) \in \mathrm{Her}_s^d \mid E_{11} \otimes \tau_1 + \ldots + E_{dd} \otimes \tau_d \succcurlyeq 0 \right\}.$$

The positivity condition is equivalent to $\tau_i \succcurlyeq 0$ for all i, which gives this free set the name *free positive orthant*. Note that for $s = 1$, the 'free' variables become real numbers,

$$C_1(p) = \left\{ (a_1, \ldots, a_d) \in \mathbb{R}^d \mid a_i \geqslant 0 \ \forall i \right\},$$

which defines the positive orthant in d dimensions.

It is easy to see that any free semialgebraic set is closed under direct sums, meaning that if $(\tau_1, \ldots, \tau_d) \in C_s(p)$, $(\chi_1, \ldots, \chi_d) \in C_r(p)$ then

$$(\tau_1 \oplus \chi_1, \ldots, \tau_d \oplus \chi_d) \in C_{r+s}(p),$$

where $\tau_i \oplus \chi_i$ denotes the block diagonal sum of two Hermitian matrices. This is because $p(\tau_1 \oplus \chi_1, \ldots \tau_d \oplus \chi_d) = p(\tau_1, \ldots \tau_d) \oplus p(\chi_1, \ldots \chi_d)$, which is psd if and only if each of the terms is psd.

Note also that a *semialgebraic set* is a Boolean combination of $C_1(p_i)$ for a finite set of polynomials p_i. A 'free semialgebraic set' is thus a noncommutative generalisation thereof, with the difference that usually a single polynomial p is considered.

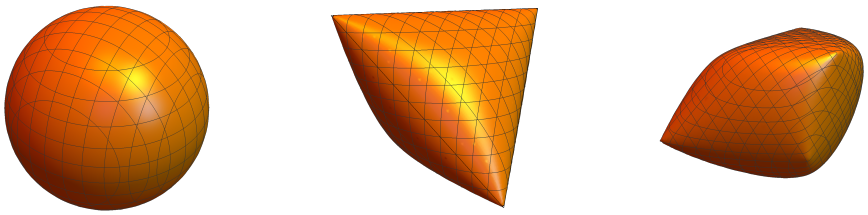

Figure 1: Some three-dimensional spectrahedra taken from [42]. Spectrahedra are convex sets described by a linear matrix inequality, and polyhedra are particular cases of spectrahedra.

A very special case of free semialgebraic sets are *free spectrahedra*, which arise from linear matrix polynomials. A *linear matrix polynomial* is a matrix polynomial where every word ω depends only on one variable, i.e.

$$\ell = \sigma_0 \otimes 1 + \sum_{i=1}^{d} \sigma_i \otimes z_i$$

with 1 being the empty word, and all $\sigma_i \in \mathrm{Her}_m$. The corresponding free set at level s is given by

$$C_s(\ell) = \left\{ (\tau_1, \ldots, \tau_d) \in \mathrm{Her}_s^d \mid \sigma_0 \otimes I_s + \sum_{i=1}^{d} \sigma_i \otimes \tau_i \succcurlyeq 0 \right\}$$

and $\mathcal{C}(\ell)$ is called a *free spectrahedron*. The first level set,

$$C_1(\ell) = \left\{ (a_1, \ldots, a_d) \in \mathbb{R}^d \mid \sigma_0 + a_1\sigma_1 + \cdots + a_d\sigma_d \succcurlyeq 0 \right\},$$

is known as a *classical spectrahedron*, or simply, a *spectrahedron* (see Fig. 1 for some three-dimensional spectrahedra). If all σ_i are diagonal in the same basis, then the spectrahedron $C_1(\ell)$ becomes a polyhedron. (Intuitively, polyhedra have flat facets whereas the borders of spectrahedra can be round, as in Fig. 1.) Thus, every polyhedron is a spectrahedron, but not vice versa.

While the linear image (i.e. the *shadow*) of a polyhedron is a polyhedron, the shadow of a spectrahedron needs not to be a spectrahedron. The forthcoming book [42] presents a comprehensive treatment of spectrahedra and their shadows.[4]

3 One wonderland through two looking glasses

Let us now explain some recent results that illustrate how concepts and methods from the two disciplines interact. We will focus on positivity and separabil-

[4] Shadows can be very different from the actual thing, as this shadow art by Kumi Yamashita shows.

ity (Section 3.1), quantum magic squares (Section 3.2), non-local games (Section 3.3), and positivity in tensor networks (Section 3.4).

3.1 Positivity and separability

For fixed $d, s \in \mathbb{N}$ consider the set of states and separable states in $\mathrm{Mat}_d \otimes \mathrm{Mat}_s$, namely $\mathrm{State}_{d,s}$ and $\mathrm{Sep}_{d,s}$, respectively. Both sets are closed in the real vector space $\mathrm{Her}_d \otimes \mathrm{Her}_s$. Moreover, both are semialgebraic, since $\mathrm{State}_{d,s}$ is a classical spectrahedron, and $\mathrm{Sep}_{d,s}$ can be proven to be semialgebraic using the *projection theorem/quantifier elimination* in the theory of real closed fields (see, e.g., [47]).

It has long been known that $\mathrm{Sep}_{d,s}$ is a strict subset of $\mathrm{State}_{d,s}$ whenever $d, s > 1$. A recent work by Fawzi [26], building on Scheiderer's [49], strengthens this result, by showing that the geometry of these two sets is significantly different:

Theorem 1 ([26]). *If $d + s > 5$ then $\mathrm{Sep}_{d,s}$ is not a spectrahedral shadow.*

Recall that a spectahedral shadow is the linear image of a spectrahedron.

Together with the relations of Table 2, it follows from the previous result that the corresponding sets of linear maps $T : \mathrm{Mat}_d \to \mathrm{Mat}_s$ satisfy that:

(i) Entanglement-breaking maps form a convex semialgebraic set which is not a spectrahedral shadow,

(ii) Completely positive maps form a spectrahedron, and

(iii) Positive maps form a convex semialgebraic set which is not a spectrahedral shadow. This follows from (i), the duality of positive maps and entanglement-breaking maps, and the fact that duals of spectrahedral shadows are also spectrahedral shadows [42].

Let us now consider the set of states and separable states as free sets. Namely, for fixed $d \geqslant 1$ let

$$\mathrm{State}_d := \big(\mathrm{State}_{d,s}\big)_{s\in\mathbb{N}} \quad \text{and} \quad \mathrm{Sep}_d := \big(\mathrm{Sep}_{d,s}\big)_{s\in\mathbb{N}}.$$

This is a particular case of the setup described above, where $V = \mathrm{Mat}_d$ and the involution is provided by $\dagger$. Moreover, both sets satisfy the condition of free convexity (Eq. (4)). In addition, State_d is a free spectrahedron, whereas Sep_d is not, since for fixed s it is not even a classical spectrahedral shadow at level s due to Theorem 1.

Viewing states as free sets also leads to an easy conceptual proof of the following result [13], which was first proven by Cariello [8].

Theorem 2 ([13, 8])**.** *For arbitrary $d, s \in \mathbb{N}$, if $\rho \in \mathrm{State}_{d,s}$ is of tensor rank 2, i.e. it can be written as*

$$\rho = \sigma_1 \otimes \tau_1 + \sigma_2 \otimes \tau_2, \tag{6}$$

where σ_i and τ_i are Hermitian, then it is separable.

Note that σ_i and τ_i need not be psd. Let us sketch the proof of [13] to illustrate the method.

Proof. Consider the linear matrix polynomial $\ell = \sigma_1 \otimes z_1 + \sigma_2 \otimes z_2$, where σ_1, σ_2 are given in Eq. (6). The fact that ρ is a state means that the corresponding free set of level s contains (τ_1, τ_2):

$$(\tau_1, \tau_2) \in C_s(\ell).$$

At level one, the spectrahedron $C_1(\ell)$ is a convex cone in $\mathbb{R}^2$. A convex cone in the plane must be a *simplex cone*, i.e. a cone whose number of extreme rays equals the dimension of the space. In $\mathbb{R}^2$ this means that the cone is spanned by two vectors,

$$C_1(\ell) = \mathrm{cone}\{v_1, v_2\},$$

where $v_1, v_2 \in \mathbb{R}^2$. When the cone at level one is a simplex cone, the free convex cone is fully determined [25, 27].

In addition, the sets

$$T_s := \{v_1 \otimes \eta_1 + v_2 \otimes \eta_2 \mid 0 \preccurlyeq \eta_i \in \mathrm{Her}_s\}$$

also give rise to a free convex cone $(T_s)_{s \in \mathbb{N}}$, and we have that $T_1 = C_1(\ell)$.

These two facts imply that $T_s = C_s(\ell)$ for all $s \in \mathbb{N}$. Using a representation for (τ_1, τ_2) in T_s, and substituting into Eq. (6) results in a separable decomposition of ρ. □

The crucial point in the proof is that when the cone at level one is a simplex cone, the free convex cone is fully determined. This is not a very deep insight — it can easily be reduced to the case of the positive orthant, where it is obvious.

Note that the separable decomposition of ρ obtained in the above proof contains only two terms — in the language of [13, 21], ρ has separable rank 2.

References [6, 7] also propose to use free spectrahedra to study some problems in quantum information theory, but from a different perspective. Given d Hermitian matrices $\sigma_1, \ldots, \sigma_d \in \mathrm{Her}_m$, one would like to know whether they fulfill

$$0 \preccurlyeq \sigma_i \preccurlyeq I_m,$$

because this implies that each σ_i gives rise to the binary POVM consisting of $\sigma_i, I_m - \sigma_i$. In addition, one would like to know whether $\sigma_1, \ldots, \sigma_d$ are *jointly measurable*, meaning that these POVMs are the marginals of one POVM (see [6] for an exact definition).

Now use $\sigma_1, \ldots, \sigma_d$ to construct the linear matrix polynomial

$$\ell := I_m \otimes 1 - \sum_{i=1}^{d} (2\sigma_i - I_m) \otimes z_i$$

and consider its free spectrahedron $\mathcal{C}(\ell) = (C_s(\ell))_{s\in\mathbb{N}}$. Define the *matrix diamond* as the free spectrahedron $\mathcal{D} = (D_s)_{s\in\mathbb{N}}$ with

$$D_s := \left\{ (\tau_1, \ldots, \tau_d) \in \mathrm{Her}_s^d \mid I_s - \sum_{i=1}^{d} \pm\tau_i \succcurlyeq 0 \right\},$$

where all possible choices of signs $\pm$ are taken into account. Note that D_1 is just the unit ball of $\mathbb{R}^d$ in 1-norm, which explains the name *diamond*. Note also that $D_1 \subseteq S_1(\ell)$ is equivalent to $0 \preccurlyeq \sigma_i \preccurlyeq I_m$ for all $i = 1, \ldots, d$. Since these finitely many conditions can be combined into a single linear matrix inequality (using diagonal blocks of matrix polynomials), $\mathcal{D}$ is indeed a free spectrahedron. The following result translates the joint measurability to the containment of free spectrahedra:

Theorem 3 ([6]). *$\sigma_1, \ldots, \sigma_d$ are jointly measurable if and only if $\mathcal{D} \subseteq \mathcal{C}(\ell)$.*

That one free spectrahedron is contained in another, $\mathcal{D} \subseteq \mathcal{C}(\ell)$, means that each of their corresponding levels satisfy the same containment, i.e. $D_s \subseteq C_s(\ell)$ for all $s \in \mathbb{N}$.

The containment of spectrahedra and free spectrahedra has received considerable attention recently [4, 31, 32, 27, 45]. One often studies *inclusion constants* for containment, which determine how much the small spectrahedron needs to be shrunk in order to obtain inclusion. In [6, 7] this is used to quantify the degree of incompatibility, and to obtain lower bounds on the joint measurability of quantum measurements.

Figure 2: (Left) The magic square on the façade of the Sagrada Família in Barcelona, where every row and column adds to 33. (Right) The magic square in Albrecht Dürer's lithograph *Melencolia I*, where every row and column adds to 34.

3.2 Quantum magic squares

Let us now look at magic squares and their quantum cousins.

A *magic square* is a $d \times d$-matrix with positive entries such that every row and column sums to the same number (see Fig. 2 for two beautiful examples.) A *doubly stochastic matrix* is a $d \times d$-matrix with real nonnegative entries, in which each row and each column sums to 1. So doubly stochastic matrices contain a probability measure in each row and each column. For example, dividing every entry of Dürer's magic square by 34 results in a doubly stochastic matrix. Now, the set of doubly stochastic matrices forms a polytope, whose vertices consist of the *permutation matrices*, i.e. doubly stochastic matrices with a single 1 in every row and column and 0 elsewhere (that is, permutations of the identity matrix). This is the content of the famous Birkhoff–von Neumann Theorem.

A 'quantum' generalization of a doubly stochastic matrix is obtained by putting a POVM (defined in Section 2.1) in each row and each column of a $d \times d$-matrix. This defines a *quantum magic square* [14]. That is, in passing from doubly stochastic matrices to quantum magic squares, we promote the nonnegative numbers to psd matrices. The normalisation conditions on the numbers (that they sum to 1) become the normalisations of the POVM (that they sum to the identity matrix).

What is a quantum generalisation of a permutation matrix? Permutation matrices only contain 0s and 1s, so in passing to the quantum version, we promote 0 and 1 to orthogonal projectors (given that 0 and 1 are the only numbers that square to themselves). The relevant notion is thus that of a *projection valued measure* (PVM), in which each measurement operator $\tau_1, \ldots, \tau_d$ is an orthogonal projection, $\tau_i^2 = \tau_i$. *Quantum permutation matrices* are magic squares containing a PVM

in each row and column [2].[5]

While PVMs are a special case of POVMs, every POVM *dilates* to a PVM (see, e.g., [46]):

Theorem 4 (Naimark's Dilation Theorem)**.** *Let $\tau_1, \ldots, \tau_d$ (of size $m \times m$) form a POVM. Then there exists a PVM $\sigma_1, \ldots, \sigma_d$ (of size $n \times n$, for some n) and a matrix $v \in \mathrm{Mat}_{n,m}$ such that*

$$v^* \sigma_i v = \tau_i \text{ for all } i = 1, \ldots, d.$$

In terms of free sets, this theorem states that *the free convex hull of the set of PVMs is precisely the set of POVMs*. Both sets are free semialgebraic, and the POVMs even form a free spectrahedron.

Through the glass of free semialgebraic geometry, quantum magic squares form a free spectrahedron over the space $V = \mathrm{Mat}_d$, equipped with entrywise complex conjugation as an involution. Level s corresponds to POVMs with matrices of size $s \times s$, and thus level 1 corresponds to doubly stochastic matrices. We thus recover the magic in the classical world at level 1, and we have an infinite tower of levels on top of that expressing the quantum case.

Furthermore, quantum permutation matrices form a free semialgebraic set whose first level consists of permutation matrices. The 'classical magic' is thus again found at level 1, and the quantum magic is expressed in an infinite tower on top of it.

Now, recall that the Birkoff–von Neumann Theorem says that the convex hull of the set of permutation matrices is the set of doubly stochastic matrices. So the permutation matrices are the vertices of the polytope of doubly stochastic matrices. In the light of the towers of quantum magic squares and quantum permutation matrices, this theorem fully characterises what happens at level one. We ask whether a similar characterisation is possible for the quantum levels: *Is the free convex hull of quantum permutation matrices equal to the set of quantum magic squares?*

This question can be phrased in terms of dilations as follows. By Naimark's Dilation Theorem we know that every POVM dilates to a PVM. The question is whether this also holds for a two-dimensional array of POVMs, i.e. whether every square of POVMs can dilated to a square of PVMs. The non-trivial part is that the dilation must work simultaneously for all POVMs in the rows and columns. The two-dimensional version of Naimark's Dilation Theorem can thus be phrased as: *Does every quantum magic square dilate to a quantum permutation matrix?*

The answer to these questions is 'no': these quantum generalisations fail to be true in the simplest nontrivial case. This means that there must exist very strange (and thus very interesting) quantum magic squares:

[5] See the closely related notion of *quantum Latin squares* [37, 35], which in essence are quantum permutation matrices with rank 1 projectors.

Theorem 5 ([14]). *For each $d \geqslant 3$, the free convex hull of the free semialgebraic set of $d \times d$ quantum permutation matrices is strictly contained in the free spectrahedron of quantum magic squares. This strict containment already appears at level $s = 2$.*

The latter statement means that there is a $d \times d$-matrix with POVMs of size 2×2 in each row and column which does not dilate to a matrix with a PVM in each row and column.

In other words, the tower of quantum levels does not admit the same kind of 'easy' characterisation as level one or the case of a single POVM — at least not the natural generalisations we have considered here. This is yet another sign of the richer structure of the quantum world compared to the classical one.

3.3 Non-local games and quantum correlations

Consider a game with two players, Alice and Bob, and a referee. The referee chooses a question randomly from finite sets Q_A and Q_B for Alice and Bob, respectively, and sends them to Alice and Bob. Upon receiving her question, Alice chooses from a finite set $\mathcal{A}_A$ of answers, and similarly Bob chooses his answer from the finite set $\mathcal{A}_B$. They send their answers to the referee, who computes a winning function

$$w\colon\ Q_A \times Q_B \times \mathcal{A}_A \times \mathcal{A}_B \to \{0,1\}$$

to determine whether they win or lose the game (value of w being 1 or 0, respectively).

During the game, Alice and Bob know both the winning function w and the probability measure on $Q_A \times Q_B$ used by the referee to choose the questions. So before the game starts Alice and Bob agree on a joint strategy. However, during the game Alice and Bob are 'in separate rooms' (or in separate galaxies) so they cannot communicate. In particular, Alice will not know Bob's question and vice versa. In order to find the strategy that maximises the winning probability, Alice and Bob have to solve an optimisation problem.[6]

What kind of strategies may Alice and Bob choose? It depends on the resources they have. First, in a *classical deterministic strategy*, both Alice and Bob reply deterministically to each of their questions, and they do so independently of each other. This is described by two functions

$$c_A\colon\ Q_A \to \mathcal{A}_A \quad \text{and} \quad c_B\colon\ Q_B \to \mathcal{A}_B,$$

which specify which answer Alice and Bob give to each question.

[6] Thus, strictly speaking, this is not a game in the game-theoretic sense, but (just) an optimisation problem.

Slightly more generally, in a *classical randomised strategy*, Alice and Bob's answers are probabilistic, but still independent of each other. This is described by

$$r_A \colon Q_A \to \Pr(\mathcal{A}_A) \quad \text{and} \quad r_B \colon Q_B \to \Pr(\mathcal{A}_B),$$

where $\Pr(S)$ denotes the set of probability measures on the set S. Namely, if Alice receives question a, the probability that she answers x is given by $r_A(a)(x)$, where $r_A(a)$ is the probability measure on $\mathcal{A}_A$ corresponding to question a. Similarly, Bob answers y to b with probability $r_B(b)(y)$. Since Alice and Bob answer independently of each other, the joint probability of answering x, y upon questions a, b is the product of the two,

$$p(x,y \mid a,b) = r_A(a)(x) \cdot r_B(b)(y). \tag{7}$$

Finally, a *quantum strategy* allows them to share a bipartite state $\rho \in \mathrm{State}_{d,s}$. The questions determine which measurement to apply to their part of the state, and the measurement outcomes determine the answers. This is described by functions

$$q_A \colon Q_A \to \mathrm{POVM}_d(\mathcal{A}_A) \quad \text{and} \quad q_B \colon Q_B \to \mathrm{POVM}_s(\mathcal{A}_B) \tag{8}$$

whose image is the set of POVMs with matrices of size $d \times d$ and $s \times s$, respectively, on the respective sets of answers. The probability that Alice answers x upon receiving a is described by $q_A(a)(x)$, which is the psd matrix that the POVM $q_A(a)$ assigns to answer x. Similarly, Bob's behaviour is modelled by $q_B(b)(y)$. Since they act independently of each other, this is described by the tensor product of the two. Using rule (1), we obtain that their joint probability is given by

$$p(x,y \mid a,b) = \mathrm{tr}\left(\rho \left(q_A(a)(x) \otimes q_B(b)(y)\right)\right). \tag{9}$$

Now, the table of conditional probabilities

$$\left(p(x,y \mid a,b)\right)_{(a,b,x,y) \in Q_A \times Q_B \times \mathcal{A}_A \times \mathcal{A}_B}$$

is called the *correlation matrix* of the respective strategy. For any given kind of strategy, the set of correlation matrices is the feasible set of the optimisation problem that Alice and Bob have to solve. The objective function of this optimisation problem is given by the winning probability. Since this objective function is linear in the correlation matrix entries, one can replace the feasible set by its convex hull.

The important fact is that quantum strategies cannot be reproduced by classical randomised strategies:

Theorem 6 ([3, 10])**.** *If at least* 2 *questions and* 2 *answers exist for both Alice and Bob, the convex hull of correlation matrices of classical randomised strategies is strictly contained in the set of correlation matrices of quantum strategies.*

For classical randomised strategies, passing to the convex hull has the physical interpretation of including a *hidden variable*. The latter is a variable whose value is unknown to us, who are describing the system, and it is usually denoted λ. However, this mysterious variable λ is shared between Alice and Bob, and it will determine the choice of their POVMs together with their respective questions a, b. This is the physical interpretation of the convex hull

$$p(x,y \mid a,b) = \sum_{\lambda} q_\lambda \, r_A(a,\lambda)(x) \cdot r_B(b,\lambda)(y),$$

where q_λ is the probability of the hidden variable taking the value λ. For example, we can imagine that Alice and Bob are listening to a radio station which plays songs from a certain list, but this is a 'private' radio station to which we have no access. The song at the moment of playing the game (i.e. receiving the questions) will determine the value of λ (i.e. λ is an index of that list).

Theorem 6 thus states that quantum strategies cannot be emulated by classical strategies, even if we take into account 'mysterious' hidden variables.

Let us now approach these results from the perspective of free sets. Assume for simplicity that all four sets $Q_A, Q_B, \mathcal{A}_A, \mathcal{A}_B$ have two elements. A quantum strategy consists of a state $\rho \in \mathrm{State}_{d,s}$ and the following psd matrices for Alice and Bob, respectively, satisfying this normalisation condition:

$$\begin{aligned} &\sigma_j^{(i)} \succcurlyeq 0 \text{ and } \tau_j^{(i)} \succcurlyeq 0 \\ &\text{such that } \sum_j \sigma_j^{(i)} = I_d \text{ and } \sum_j \tau_j^{(i)} = I_s, \end{aligned} \tag{10}$$

where $i, j = 1, 2$. The superscript refers to the questions and the subscript to the answers. The correlation matrix is given by

$$\left(\mathrm{tr}\left(\rho \, \left(\sigma_k^{(i)} \otimes \tau_l^{(j)} \right) \right) \right)_{i,j,k,l}.$$

Using the spectral decomposition of $\rho = \sum_r v_r v_r^*$, it can be written as

$$\left(\sum_r v_r^* \left(\sigma_k^{(i)} \otimes \tau_l^{(j)} \right) v_r \right)_{i,j,k,l}, \tag{11}$$

where $v_r \in \mathbb{C}^d \otimes \mathbb{C}^s$ and $\sum_r v_r^* v_r = 1$. Through the looking glass of free semialgebraic geometry, this is first level of a free convex hull. To see this, define the free

set I as

$$I = \bigcup_{d,s\geq 1} \left\{ \left(\sigma_k^{(i)} \otimes \tau_l^{(j)} \right)_{i,j,k,l} \in \mathrm{Mat}_4(\mathrm{Mat}_d \otimes \mathrm{Mat}_s) \mid \sigma_k^{(i)} \text{and } \tau_l^{(j)} \text{satisfy (10)} \right\} \tag{12}$$

(Note that the 4 is due the fact that we have 2 questions and 2 answers; more generally we would have a matrix of size $|Q_A||Q_B| \times |\mathcal{A}_A||\mathcal{A}_B|$. Note also that the ordering of questions and answers of Alice and Bob is irrelevant for the following discussion.)

If we look at level 1 of this free set, we encounter that I_1 is the subset of $\mathrm{Mat}_4(\mathbb{R})$ consisting precisely of the correlation matrices of classical randomized strategies. In other words, when $d = s = 1$, the formula coincides with that of (7). Furthermore, higher levels of this free set contain the tensor products of POVMs of Alice and Bob in the corresponding space Mat_d and Mat_s. Since I_1 is called the *independence model* in algebraic statistics [23], we call I the *free independence model*, since this is the natural noncommutative generalisation of independent strategies.

Let us now consider the free convex hull of I. First of all, computing the conditional probabilities of a pair of POVMs with a given state ρ corresponds to compressing to level 1 with the vectors $\{v_r\}$ given by the spectral decomposition of ρ, as in (11). So *the set of quantum correlations is the first level of the free convex hull of the free independence model.*

We thus encounter an interesting phenomenon: the free convex hull of a free set can be larger than the classical convex hull at a fixed level. Specifically, the convex hull of I_1 is the set of classical correlations, whereas the free convex hull of I at level 1 is the set of quantum correlations, which are different by Theorem 6. In fact, wilder things can happen: fractal sets can arise in the free convex hull of free semialgebraic sets [1]. We wonder what these results imply for the corresponding quantum information setup.

Now, in the free convex hull of I, what do higher levels correspond to? Compressing to lower levels (i.e. with smaller ds) corresponds to taking the partial trace with a psd matrix of size smaller than ds. This results in 4 psd matrices (one for each i, j, k, l), each of size $< ds$, which do not need to be an elementary tensor product.

What about 'compressing' to higher levels? Any compression to a higher level can be achieved by direct sums of the POVMs of Alice and Bob and a compression to a lower level as we just described. The number of elements in this direct sum is precisely n in (4). Another way of seeing that the direct sum is needed is by noting that, if $n = 1$, the matrices v_i cannot fulfill the normalisation condition on the right hand side of (4). In quantum information terms, this says that a POVM in a given dimension cannot be transformed to a POVM in a larger dimension by means of an isometry, because the terms will sum to a projector instead of the identity.

Let us make two final remarks. The first one is that I is not a free semialgebraic set, for the simple reason that it is not closed under direct sums (which is a property of these sets, as we saw in Section 2.2), as is easily checked.

The second remark is that the free convex hull of the free independence model is not closed. This follows from the fact that, at level 1, this free convex hull fails to be closed, as shown in [51] and for smaller sizes in [24].

Theorem 7 ([51, 24])**.** *For at least* 5 *questions and* 2 *answers, the set of quantum correlation matrices is not closed.*

In our language, this implies that the level ds — which is to be compressed to level 1 in the construction of the free convex hull — cannot be upper bounded. That is, the higher ds, the more things we will obtain in its compression to level 1.

In the recent preprint [28] the membership problem in the closure of the set of quantum correlations is shown to be undecidable, for a fixed (and large enough) size of the sets of questions and answers.

A computational approach to quantum correlations, comparable to sums-of-squares and moment relaxation approaches in polynomial optimisation [5, 42], is the *NPA hierarchy* [38, 39, 40]. We briefly describe the approach here, omitting technical details. Assume one is given a table

$$p = (p(x,y \mid a,b))_{(a,b,x,y)\in Q_A\times Q_B\times \mathcal{A}_A\times \mathcal{A}_B},$$

and the task is to check whether it is the correlation matrix of a quantum strategy. The NPA hierarchy provides a family of necessary conditions, each more stringent than the previous one, for p to be a quantum strategy.

In order to understand the NPA hierarchy, we will first *assume* that p is a correlation matrix, i.e. there is a state ρ and strategies such that (9) holds. We will use this state and strategies to define a positive functional on a certain algebra. Namely, we consider the *game $*$-algebra*

$$\mathcal{G} := \mathbb{C}\langle Q_A \times \mathcal{A}_A, Q_B \times \mathcal{A}_B\rangle.$$

This is an algebra of polynomials in certain noncommuting variables. Explicitly, for each question and answer pair from Alice and Bob, (a,x) and (b,y), there is an associated self-adjoint variable, $z_{(a,x)}$ and $z_{(b,y)}$, respectively. $\mathcal{G}$ consists of all polynomials with complex coefficients in these variables; for example, the

monomial $z_{(a,x)}z_{(b,y)} \in \mathcal{G}$. Now, *if we had the strategy* ρ, q_A, q_B we could construct a linear functional

$$\varphi\colon\ \mathcal{G} \to \mathbb{C}$$

by evaluating the variables $z_{(a,x)}$ and $z_{(b,y)}$ at the psd matrices $q_A(a)(x) \otimes I_s$ and $I_d \otimes q_B(b)(y)$, respectively, and computing the trace inner product with the state ρ. So, in particular, evaluating φ at the monomial $z_{(a,x)}z_{(b,y)}$ would yield

$$\varphi(z_{(a,x)}z_{(b,y)}) = \operatorname{tr}\left(\rho\left((q_A(a)(x) \otimes I_s) \cdot (I_d \otimes q_B(b)(y))\right)\right) \tag{13}$$

$$= p(x,y \mid a,b). \tag{14}$$

The crucial point is that φ evaluated at this monomial needs to have the value $p(x,y \mid a,b)$ *for any strategy* realising p. In other words, the linear constraint on φ expressed in Equation (14) must hold even if we do not know the strategy. This functional must satisfy other nice properties independently of the strategy too, such as being positive.

This perspective is precisely the one we now take. Namely, we assume that the strategy ρ, q_A, q_B is *not* given (since our question is whether p is a quantum strategy at all), and we search for a functional on $\mathcal{G}$ that has the stated properties (or other properties, depending on the kind of strategies one is looking for). When restricted to a finite-dimensional subspace of $\mathcal{G}$, this becomes a semidefinite optimisation problem, as can be easily checked. The dimension of this subspace will be the parameter indicating the level of the hierarchy, which is gradually increased. Solvability of all these semidefinite problems is thus a necessary condition for p to be a quantum correlation matrix. In words, the levels of the NPA hierarchy form an outer approximation to the set of correlations. Conversely, if all/many of these problems are feasible, one can apply a (truncated) *Gelfand-Naimark-Segal (GNS) construction* (see for example [46]) to the obtained functional, and thereby try to construct a quantum strategy that realises p. This is the content of the NPA hierarchy from the perspective of free semialgebraic geometry.

3.4 Positivity in tensor networks

Let us finally explain some results about positivity in tensor networks. The results are not as much related to free semialgebraic geometry as to positivity and sums of squares, as we will see.

Since the state space of a composite quantum system is given by the tensor product of smaller state spaces (Eq. (2)), the global dimension d grows exponentially with the number of subsystems n. Very soon it becomes infeasible to work with the entire space — to describe $n = 270$ qubits $d_i = 2$, we would need to deal with a space dimension $d \sim 2^{270} \sim 10^{80}$, the estimated number of atoms in the Universe. To describe anything at the macro-scale involving a mole of particles, $\sim 10^{23}$, we

would need a space dimension of $\sim 2^{10^{23}}$, which is much larger than a googol (10^{100}), but smaller than a googolplex ($10^{10^{100}}$). These absurd numbers illustrate how quickly the Hilbert space description becomes impractical — in practice, it works well for a few tens of qubits.[7]

Fortunately, many physically relevant states admit an efficient description. The ultimate reason is that physical interactions are local (w.r.t. a particular tensor product decomposition; this decomposition typically reflects spatial locality). The resulting relevant states admit a description using only a few terms for every local Hilbert space. The main idea of tensor networks is precisely to use a few matrices for every local Hilbert space Mat_{d_i} (Eq. (2); see, e.g., [44, 9]).

Now, this idea interacts with *positivity* in a very interesting way. Positivity is a property in the global space Mat_d which cannot be easily translated to positivity properties in the local spaces. As we will see, there is a 'tension' between using a few matrices for each local Hilbert space and representing the positivity locally. This mathematical interplay has implications for the description of quantum many-body systems, among others.

Let us see one example of a tensor network decomposition where this *positivity problem* appears. To describe a mixed state in one spatial dimension with periodic boundary conditions we use the *matrix product density operator form* (MPDO) of ρ,

$$\rho = \sum_{i_1,\ldots,i_n=1}^{r} \rho^{(1)}_{i_1,i_2} \otimes \rho^{(2)}_{i_2,i_3} \otimes \cdots \otimes \rho^{(n)}_{i_n,i_1}.$$

The smallest such r is called the *operator Schmidt rank* of ρ [52, 55]. Clearly, every state admits an MPDO form, and the ones with small r can be handled efficiently. But how is the positivity of ρ reflected in the local matrices? Clearly, if all local matrices are psd (i.e. $\rho^{(k)}_{ij} \succcurlyeq 0$) then ρ will be psd. But some sums of non-psd matrices will also give rise to a global psd matrix, since negative subspaces may cancel in the sum. Can one easily characterise the set of local matrices whose sum is psd? The short answer is 'no'.

For further reference, if all local matrices are psd, so that ρ is separable, the corresponding r is called the *separable rank* of ρ [21, 17].

To obtain a local certificate of positivity, we first express $\rho = \xi\xi^*$ (which is possible only if ρ is psd) and then apply the tensor network 'philosophy' to ξ, i.e. express ξ as an MPDO:

$$\rho = \xi\xi^* \quad \text{with} \quad \xi = \sum_{i_1,\ldots,i_n=1}^{r} \xi^{(1)}_{i_1,i_2} \otimes \xi^{(2)}_{i_2,i_3} \otimes \cdots \otimes \xi^{(n)}_{i_n,i_1}.$$

[7] The lack of scalability of this description is far from being a unique case in physics — most theories are not scalable. One needs to find the new relevant degrees of freedom at the new scale, which will define an emergent theory.

This is the *local purification form* of ρ. Note that there are many ξ that satisfy $\rho = \xi\xi^*$, as ξ needs not to be Hermitian or a square matrix (it could be a column vector). The smallest r among all such ξ is called the *purification rank* of ρ.

The interesting point for the purposes of this paper is that the purification rank is a *noncommutative generalisation* of the positive semidefinite rank of a nonnegative matrix. There are many more such connections: the separable rank, the translational invariant (t.i.) purification rank, and the t.i. separable rank are noncommutative generalisations of the nonnegative rank, the cpsd rank and the cp rank of nonnegative matrices, respectively [21]. As a matter of fact, this connection holds in much greater generality, as we will explain below. In all of these cases, the ranks coincide for quantum states that are diagonal in the computational basis.

From our perspective, this connection is beneficial for both sides. For example, for quantum many-body systems, this insight together with the results by [30] leads to the following result:

Theorem 8 ([22, 21])**.** *The purification rank cannot be upper bounded by a function of the operator Schmidt rank only. The separable rank cannot be upper bounded by a function of the purification rank only.*

(It is worth noting these separations are not robust, as they disappear in the approximate case for certain norms [19].)

Conversely, the quantum perspective provides a natural and well-motivated path for generalisation of the 'commutative' results about cpsd rank, cp rank, etc. For example, in [30] it is shown that the extension complexity of a polytope w.r.t. a given cone is given by the rank of the slack matrix of that polytope w.r.t. that cone. We wonder whether this result could be a generalisation to the noncommutative world. This would give a *geometric* interpretation of the purification rank, the separable rank and their symmetric versions, perhaps as extension complexities of some objects.

Symmetry is a central property in physics, both conceptually and practically. Conceptually, symmetry is the other side of the coin of a conserved quantity (by Noether's Theorem). Practically, it allows for more efficient mathematical descriptions, as symmetric objects have fewer degrees of freedom. For example, in the above context, ρ is *translational invariant* if it remains unchanged under cyclic permutations of the local systems. This raises the question: is there an MPDO form that explicitly expresses this symmetry? For example, the following form does,

$$\rho = \sum_{i_1,\ldots,i_n=1}^{r} \rho_{i_1,i_2} \otimes \rho_{i_2,i_3} \otimes \cdots \otimes \rho_{i_n,i_1},$$

because it uses the same matrices on every site, and the arrangement of indices is such that a cyclic permutation of the local systems does not change ρ. But does this hold for other symmetries too?

The existence of such *invariant decompositions* and their corresponding ranks has been studied in a very general framework [17]. Explicitly, every tensor decomposition is represented as a simplicial complex, where the individual tensor product spaces are associated to the vertices, and the summation indices to the facets. The symmetry is modelled by a group action on the simplicial complex. The central result is that an invariant decomposition exists if the group action is *free* on the simplicial complex [17]. Just to give one example, if $\rho \in \mathrm{Mat}_d \otimes \mathrm{Mat}_d$ is separable and symmetric, it will in general *not* admit a decomposition of the type

$$\rho = \sum_\alpha \rho_\alpha \otimes \rho_\alpha \quad \text{with all } \rho_\alpha \text{ psd},$$

but it will have one of the type

$$\rho = \sum_{\alpha,\beta} \rho_{\alpha,\beta} \otimes \rho_{\beta,\alpha} \quad \text{with all } \rho_{\alpha,\beta} \text{ psd}.$$

From the perspective of our framework, this is due to the fact that the group permuting the two end points of an edge does not act freely on the edge connecting them. But this group action can be made free if the two points are connected by two edges, leading to the two indices α, β in the above sum. This is one example of a *refinement* of a simplicial complex, which makes the action of the group free [17].

Finally, we remark that this framework of tensor decompositions with invariance can not only be applied to quantum-many body systems, but to any object in a tensor product space. One example are multivariate symmetric polynomials with positivity conditions [20].

A related question is the existence of invariant decompositions *uniform* in the system size. Namely, given a tensor $\rho = \left(\rho_{\alpha,\beta}\right)_{\alpha,\beta=1,\ldots,r}$ with all $\rho_{\alpha,\beta} \in \mathrm{Mat}_d$, define

$$\tau_n(\rho) := \sum_{i_1,\ldots,i_n=1}^{r} \rho_{\alpha_1,\alpha_2} \otimes \rho_{\alpha_2,\alpha_3} \otimes \cdots \otimes \rho_{\alpha_n,\alpha_1} \in \mathrm{Mat}_{d^n}$$

for all $n \in \mathbb{N}$. The result, in this case, is very different from the fixed n case:

Theorem 9 ([12])**.** *Let $d, r \geqslant 7$. Then it is undecidable whether $\tau_n(\rho) \succcurlyeq 0$ for all $n \in \mathbb{N}$.*

Using this result it can be shown that a translationally invariant local purification of $\tau_n(\rho)$ *uniform in the system size* needs not to exist [12].

The proof of this theorem uses a reduction from the matrix mortality problem. In the latter, given a finite set of matrices $M_\alpha \in \mathrm{Mat}_d(\mathbb{Z})$, one is asked whether there is a word w such that $0 = M_{w_1} \cdots M_{w_n} \in \mathrm{Mat}_d(\mathbb{Z})$. While this problem is noncommutative (because matrix multiplication is), the problem about $\tau_n(\rho)$ is 'more' noncommutative. Intuitively, if all $\rho_{\alpha,\beta}$ are diagonal, we recover a version of the matrix mortality problem. Note also that the space where $\tau_n(\rho)$ lives grows with n, in contrast to the matrix mortality problem.

The decidability of a similar problem can be studied for more general algebras. In that case, $\rho_{\alpha,\beta}$ is in a certain algebra, and $\tau_n(\rho)$ is asked to be in a certain cone [16].

Let us finally explain a computational approach for the finite case. So consider n fixed and recall that after specifying some local matrices $\rho_i^{(j)}$, one wants to know whether

$$\rho = \sum_i \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(n)}$$

is psd. Since the n is fixed, this problem is decidable, but computing and diagonalising ρ is impossible for large values of n (in fact it is NP-hard [36]). So one has to come up with a different idea. What can be computed are certain *moments* of ρ, i.e. the numbers $\mathrm{tr}(\rho^k)$ for small enough k. This follows from the observation that the moments only require local matrix products,

$$\mathrm{tr}(\rho^k) = \sum_{i_1,\ldots,i_k=1}^{r} \mathrm{tr}\left(\rho_{i_1}^{(1)} \cdots \rho_{i_k}^{(1)}\right) \cdots \mathrm{tr}\left(\rho_{i_1}^{(n)} \cdots \rho_{i_k}^{(n)}\right).$$

These few moments can then be used to compute optimal upper and lower bounds on the distance of ρ to the cone of psd matrices [15]. Specifically, to compute this distance it suffices to compare ρ with $f(\rho)$, where $f\colon \mathbb{R} \to \mathbb{R}$ is the function that leaves the positive numbers unchanged, and sets the negative numbers to zero. We then approximate f by polynomial functions q of low degree, so that $\mathrm{tr}(q(\rho))$ only uses a few moments of ρ. The best results were obtained with certain *sums of squares approximations*, which can be computed with a linear or semidefinite optimisation.

4 Closing words

We have illustrated how quantum information theory and free semialgebraic geometry often study very similar mathematical objects from different perspectives.

We have given the examples of positivity and separability (Section 3.1), quantum magic squares (Section 3.2), non-local games (Section 3.3), and positivity in tensor networks (Section 3.4). In all of these cases, we have tried to illustrate how results can be transferred among the two fields, and how this can be beneficial for the two perspectives. As mentioned in the introduction, there are many similar such connections which have not been covered here.

Going back to New Hampshire's motto, we conclude that it is undecidable to determine whether to live free or die, because both the question of whether matrices generate a free semigroup and the matrix mortality problem are undecidable.

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Sparse random planar graphs

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This is a survey article on the research area of sparse random planar graphs. We first review some of the important results from the classical theory of random graphs and then consider the so–called core–kernel approach. This method allows to analyse random planar graphs via the Pólya urn model, Prüfer sequences, and the balls–into–bins model. Finally, we discuss some recent results on longest and shortest cycles and the maximum degree in sparse random planar graphs.

1 Introduction

The starting point of the history of random graphs was in 1959 when Erdős and Rényi [12, 13] introduced the so–called *Erdős–Rényi random graph* $G(n,m)$, which is a graph chosen uniformly at random from the class of all vertex–labelled graphs on vertex set $[n] := \{1,\dots,n\}$ with $m = m(n)$ edges. Since then, asymptotic properties of $G(n,m)$, i.e. when $n \to \infty$, were extensively studied (see e.g. [5, 14, 21] for an overview). A particular remarkable feature of this model is the '*threshold*' behaviour of many graph parameters. That is, certain graph parameters in $G(n,m)$ undergo a dramatic change when the number of edges m is around a 'critical' value. Perhaps the most prominent example of such a threshold phenomenon is that of the number of vertices in the largest component of $G(n,m)$, denoted by $v(L_1)$. Letting $m = d \cdot n/2$ for a constant $d > 0$, i.e. the average degree in $G(n,m)$ is d, Erdős and Rényi [13] showed that *with high probability* (meaning with probability tending to 1 as n tends to infinity, *whp* for short) the following holds in $G(n,m)$:

$$v(L_1) = \begin{cases} O(\log n) & \text{if } d < 1 \\ \Theta\left(n^{2/3}\right) & \text{if } d = 1 \\ \Theta(n) & \text{if } d > 1. \end{cases} \tag{15}$$

In other words, the component structure of $G(n,m)$ changes dramatically when $m \sim n/2$.

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Another behaviour which has been observed in $G(n,m)$ is the '*concentration*' of many graph parameters, i.e. certain graph parameters lie whp in 'small' intervals, which only depend on n and m. For instance, Łuczak [30] considered the chromatic number $\chi(G(n,m))$ for the case that $m \leq n^{7/6-\varepsilon}$ for some $\varepsilon > 0$. He proved that there exists a $u = u(n) \in \mathbb{N}$ such that whp $\chi(G(n,m)) \in \{u, u+1\}$, i.e. whp $\chi(G(n,m))$ is concentrated at two values.

Another well–studied model is the *binomial random graph* $G(n,p)$, introduced by Gilbert [16]. That is a graph on vertex set $[n]$ where each pair of vertices is connected independently by an edge with probability $p = p(n)$. It is well known that $G(n,p)$ behaves asymptotically very similarly like $G(n,m)$, when $p = m/\binom{n}{2}$ (see [14, Section 1.1] for details). For most applications one can work in whichever model is most convenient and use some standard technique to transfer results between them.

In the last decades random planar graphs [9, 11, 15, 17, 23, 34, 35, 37] and, more generally, random graphs on surfaces [8, 10, 26, 33] have attracted attention. Many exciting results have been obtained in the so–called n–vertex model, which is a graph $P(n)$ chosen uniformly at random from the class of all planar graphs on vertex set $[n]$. For example, it was shown that the maximum degree [11, 34] and the diameter [9] of $P(n)$ are strongly concentrated.

A more natural generalisation of $G(n,m)$ seems to be the *random planar graph* $P(n,m)$. That is a graph chosen uniformly at random from the class of all planar graphs on vertex set $[n]$ with $m = m(n)$ edges. Kang and Łuczak [23] and later Kang, Moßhammer, and Sprüssel [26] showed that the component structure of $P(n,m)$ undergoes a phase transition when $m \sim n/2$, similar to $G(n,m)$ (cf. (15)). Moreover, there is a second critical range at $m \sim n$, when the number of vertices outside the largest component $n - v(L_1)$ drops from linear to sublinear. Such a behaviour has not been observed in $G(n,m)$, where whp $n - v(L_1) = \Theta(n)$ as long as $m = O(n)$. According to that result, random planar graphs have been separately studied for the 'sparse' regime when $m \leq n + o(n)$ [23, 26] and the 'dense' case $m = d \cdot n/2$ where $d = d(n)$ tends to a constant in $(2,6)$ [8, 17]. In view of concentration and threshold results the sparse regime is more interesting, which is why we focus on sparse random planar graphs in this article.

Britikov [7] proved that the probability that $G(n,m)$ is planar is bounded away from zero as long as $m \leq n/2 + O\left(n^{2/3}\right)$. Thus, in this regime each property that holds whp in $G(n,m)$ is also true whp in $P(n,m)$. However, if $m = n/2 + \omega\left(n^{2/3}\right)$, the probability that $G(n,m)$ is planar tends to zero and hardly any asymptotic result of $G(n,m)$ can be transferred to $P(n,m)$. Therefore, for many graph parameters in $P(n,m)$ it is not yet known whether they exhibit a threshold or concentration behaviour in this regime. A promising tool for investigating these open problems seems to be the so–called *core–kernel approach*, which is based on a decomposition and construction technique of graphs introduced by Bollobás [4]

and Łuczak [28]. This method was recently used to obtain results on longest and shortest cycles [24] and the maximum degree [25] in sparse random planar graphs.

The rest of the article is based on [24, 25] and provides an overview of these two papers. After introducing some necessary definitions and notations in Section 1.1, we study the core–kernel approach in more detail in Section 2. Section 3 is devoted to random cores and longest and shortest cycles in random graphs. Finally in Section 4, we consider random complex parts and discuss concentration results on the maximum degree in random graphs.

1.1 Preliminaries

Throughout the article, we only consider undirected graphs or multigraphs. Moreover, we assume that all graphs are vertex–labelled, i.e. the vertex set of a graph with n vertices is $[n] = \{1, \dots, n\}$. For a fixed graph H we denote by $v(H)$ the number of vertices in H, by $e(H)$ the number of edges in H, and by $d_H(v)$ the degree of a vertex v in H. In addition, we let $\Delta(H)$ be the maximum degree in H.

To express asymptotic orders of graph parameters, we will use the standard Landau notation and the following notation from [20]. Moreover, all asymptotics are taken with respect to n, i.e. when $n \to \infty$.

Definition 1. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables and $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$. We write $X_n = \Theta_p(f)$ if for all $\delta > 0$ there exist $c_1, c_2 > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$ we have $c_1 f(n) \leq |X_n| \leq c_2 f(n)$ with a probability of at least $1 - \delta$.*

We note that $X_n = \Theta_p(f)$ if and only if whp $X_n = \Omega(f/h)$ and $X_n = O(h \cdot f)$ for each function $h = h(n) = \omega(1)$. In contrast, we have whp $X_n = \Theta(f)$ if and only if whp $X_n = \Omega(f)$ and $X_n = O(f)$. Thus, $X_n = \Theta_p(f)$ is a slightly weaker condition than whp $X_n = \Theta(f)$.

2 Core–kernel approach

Throughout the section, let $P = P(n,m)$ be the random planar graph. The main idea of the core–kernel approach is to construct another random planar graph $\tilde{P}$, which 'behaves' similarly like P, but is easier to analyse. More precisely, we will construct $\tilde{P}$ stepwise by using basic operations, like subdividing edges or replacing vertices by rooted trees. Each construction step can be investigated independently of the others, which seems to be much easier than considering P directly. Later we will see that many asymptotic results on $\tilde{P}$ can be transferred to P. In order to make that idea more precise, we introduce a decomposition of graphs into smaller parts. Later we will obtain the desired construction of $\tilde{P}$ by reversing this decomposition.

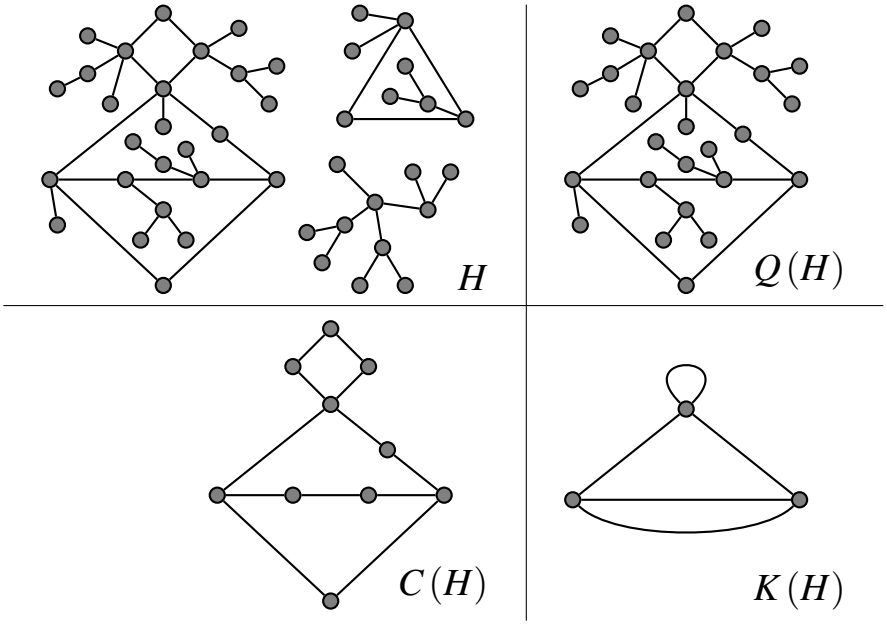

Figure 1: A graph H, its complex part $Q(H)$, core $C(H)$, and kernel $K(H)$.

Given a fixed graph H we extract the *complex part* $Q(H)$, which is the union of all components with at least two cycles. In other words, we obtain the complex part $Q(H)$ by deleting all components of H which have no or one cycle. We call components of H which are in the complex part $Q(H)$, i.e. components with at least two cycles, *complex*. Next, we delete recursively vertices of degree one in $Q(H)$ to get the *core* $C(H)$. More formally, the core $C(H)$ is the maximal subgraph of minimum degree at least two of $Q(H)$. Finally, we replace certain paths in $C(H)$ by edges to obtain the *kernel* $K(H)$. More precisely, we consider paths in $C(H)$ which consist of internal vertices of degree two and end vertices of degree at least three. This 'path' can also be a cycle having exactly one vertex of degree at least three. In order to obtain the kernel $K(H)$, we replace each such path by an edge between the two endpoints of the path. In case of a cycle, this new added edge is a loop. We note that multiple edges can be constructed, so in general the kernel $K(H)$ is a multigraph. The difference between the number of vertices in the core and the kernel is called the *subdivision number* $S(H) := v(C(H)) - v(K(H))$. We refer to Figure 1 for an illustration of this decomposition.

Reversing the above decomposition, we can construct planar graphs. We observe that a graph is planar if and only if its kernel is. Thus, we can use planar kernels, i.e. planar multigraphs with minimum degree at least three, as base cases for our construction. Given a kernel K we obtain a core C by subdividing the edges with additional vertices, i.e. we replace edges by paths. As the core is a simple graph, we need to subdivide the edges in such a way that all loops and multiple edges are

destroyed. Next, we get a complex part Q by adding recursively vertices of degree one. Equivalently, we can replace each vertex in C by a rooted tree. Finally, we obtain a graph H by adding a graph without complex components, i.e. we add components having no or one cycle.

In the next step we use this construction to create the random graph $\tilde{P}$. To that end, we introduce the following random graphs.

Definition 2. *Given a kernel K and a subdivision number $k \in \mathbb{N}$, we denote by $C(K,k)$ a graph chosen uniformly at random from the class of all cores with kernel K and subdivision number k, i.e. we obtain $C(K,k)$ by subdividing the edges of K randomly with k additional vertices. Similarly, for a fixed core C and $q \in \mathbb{N}$ we let $Q(C,q)$ be a graph chosen uniformly at random from the class of all complex parts with core C and q vertices, i.e. we replace each vertex of C randomly by a rooted tree such that we obtain a complex part with q vertices. We call $C(K,k)$ a random core (with kernel K and subdivision number k) and $Q(C,q)$ a random complex part (with core C and q vertices).*

Given numbers $n_K, m_K, k, n_Q \in \mathbb{N}$, we want to choose the planar graph $\tilde{P}$ randomly such that the kernel has n_K vertices and m_K edges, the subdivision number is k, and the number of vertices in the complex part is n_Q. This motivates the following construction. We start by choosing randomly a planar kernel K with n_K vertices and m_K edges. Given K we pick a random core $C = C(K,k)$ and then a random complex part $Q = Q(C,n_Q)$. Finally, we add to Q a random graph without complex components such that we obtain a graph $\tilde{P}$ with n vertices and m edges.

Crudely stated, the following relation between P and $\tilde{P}$ is proven in [24] for each graph property $\mathcal{R}$. If for all 'reasonable' choices of $n_K, m_K, k, n_Q \in \mathbb{N}$ whp $\mathcal{R}$ holds in $\tilde{P}$, then whp $\mathcal{R}$ is true in P. For example, if whp the kernel $K(P)$ fulfils $e(K(P)) = 3/2v(K(P))$, then 'reasonable' choices of n_K and m_K are only those which satisfy $m_K = 3/2n_K$. We note that Kang, Moßhammer, and Sprüssel [26] proved lots of results on the internal structure of P, e.g. asymptotic orders of $v(K(P))$, $e(K(P))$, $S(P)$, and $v(Q(P))$. All this information can be translated to properties which need to be fulfilled by 'reasonable' choices of n_K, m_K, k, and n_Q.

The key to find strong results on $\tilde{P}$ is a good analysis of the single construction steps. In particular, we aim to find properties of the random core $C(K,k)$ and the random complex part $Q(C,q)$ for fixed values of K, k, C, and q.

In Sections 3 we sketch the main ideas of a study of the random core $C(K,k)$ by means of the famous *Pólya urn model* from [24]. Similarly, in Section 4 we summarise an analysis of the random complex part $Q(C,q)$ via Prüfer sequences and the balls–into–bins model from [25].

3 Random core

Let K be a kernel on vertex set $[v(K)]$ and $k \in \mathbb{N}$ a subdivision number. Recall that the random core $C(K,k)$ is a graph chosen uniformly at random from the class of all cores with kernel K and subdivision number k (cf. Definition 2). In Section 3.1 we consider a simple algorithm for constructing $C(K,k)$, which allows to find a connection between $C(K,k)$ and the famous Pólya urn model (see e.g. [22, 31] for an overview of Pólya urns). In [24] this connection was used to obtain strong results on longest and shortest cycles in $P(n,m)$, which we state in Section 3.2.

3.1 Construction of the random core and the Pólya urn model

We recall that $C(K,k)$ can be obtained by subdividing the edges of K randomly with k additional vertices $v(K)+1,\dots,v(K)+k$ such that all loops and multiple edges of K get destroyed. More precisely, we perform the following stepwise construction. We start with $K_0 := K$ and let $i \in [k]$. Given the multigraph K_{i-1} we choose an edge e_i of K_{i-1} uniformly at random and subdivide e_i with the new vertex $v(K)+i$ to obtain K_i. We note that each edge of K_i corresponds to an edge e of K, i.e. it is either e or was created by subdividing e. After k steps we end up with a multigraph K_k having K as its kernel and $v(K)+k$ many vertices. If this multigraph is simple, we set $C(K,k) = K_k$ and otherwise we repeat that procedure until we obtain a simple graph K_k.

It is straightforward to check that each core with kernel K and subdivision number k is obtained equally likely, i.e. this construction creates indeed $C(K,k)$. In most of our cases the subdivision number k will be much larger than $e(K)$. Then it should be quite likely that K_k is simple. Therefore, if we study asymptotic properties of $C(K,k)$, we can assume that we always end up with a simple graph K_k already after the first round of construction.

For many applications we are only interested in the number of vertices which subdivide an edge e of K rather than the precise labels of these vertices. We call this number the subdivision number $S(e)$ of an edge e in K. At the beginning of the construction we have $S(e) = 0$ for each edge e. In step i the number $S(e)$ increases by one if the chosen edge e_i corresponds to the original edge e and otherwise $S(e)$ remains the same. Moreover, the number of edges in K_i corresponding to e is $S(e)+1$. Therefore, the probability that $S(e)$ gets increased by one in step $i+1$ is $(S(e)+1)/(e(K)+i)$. Hence, we can keep track of the subdivision numbers by a Pólya urn model (see also Figure 2). We start with $e(K)$ many balls of $e(K)$ different colours, each of them representing an edge of K, in a urn. Then in each of k steps we draw a ball uniformly at random and return it together with an additional ball of the same colour. The event that we draw a ball with a colour representing the edge e imitates a choice of an edge in K_i which corresponds to e. At the end there are $S(e)+1$ many balls of the colour representing e in the urn.

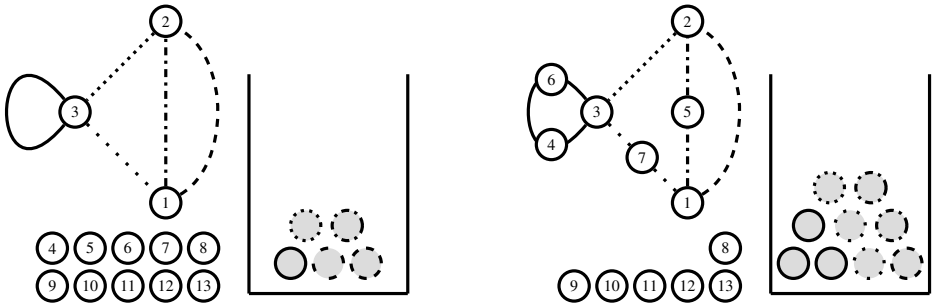

Figure 2: Construction of a random core together with the corresponding Pólya urn model. The left–hand side represents the situation at the beginning, the right-hand side after four drawings.

To sum up, if we are interested in the subdivision numbers, it suffices to consider a Pólya urn instead of looking at the precise construction of $C(K,k)$. Therefore, many asymptotic results on the Pólya urn model translate to properties of $C(K,k)$. In [24] the maximum and minimum number of balls of the same colour in the Pólya urn are analysed and thereby bounds on the longest and shortest cycles in a random planar graph are obtained. In Section 3.2 we state these results and point out differences to the cycle structure of the Erdős–Rényi random graph $G(n,m)$.

3.2 Longest and shortest cycles in random graphs

In the past many results on the cycle structure of the Erdős–Rényi random graph $G(n,m)$ were obtained [1, 6, 19, 27, 28, 29, 39]. Ajtai, Komlós, and Szemerédi [1] showed that whp $G(n,m)$ has a cycle of length $\Theta(n)$ if $m = d \cdot n/2$ for a constant $d > 1$. In contrast, Bollobás [5, Corollary 5.8] proved that whp all cycles are of constant length if $d < 1$. Moreover, relations between cycles and the component structure were investigated and questions of the following type considered.

(Q1) Is the longest cycle contained in the largest component?

(Q2) Is there even a 'threshold' f such that all cycles longer than f lie inside the largest component and all others outside?

Questions (Q1) and (Q2) are particular interesting when m is sufficiently large so that $G(n,m)$ has many cycles, but still small enough that the graph without the largest component contains some cycles. One can show that this is 'true' when $m = n/2 + s$ for $s = o(n)$ such that $s^3n^2 \to \infty$. Therefore, we restrict our attention to that case in this section.

In order to study (Q1) and (Q2) in more detail, we introduce some notations. Given a graph H, we denote by $c(H)$ the length of the longest cycle of H and by

$g(H)$ the length of the shortest cycle of H. We call $c(H)$ and $g(H)$ the *circumference* and *girth* of H, respectively. Moreover, we denote by $L_1 = L_1(H)$ the largest component of H and by $R = R(H) := H \setminus L_1(H)$ the 'rest' of H. To answer (Q1) and (Q2), it suffices to determine the length of the longest cycle inside the largest component $c(L_1)$, of the shortest cycle inside the largest component $g(L_1)$, and of the longest cycle outside the largest component $c(R)$.

Łuczak [28] determined precise asymptotic orders of $c(L_1)$, $g(L_1)$, and $c(R)$ in $G(n,m)$ (see Theorem 1). In particular, his results imply that whp the longest cycle lies inside the largest component and that $f(n) = ns^{-1}$ is a 'threshold' in the sense that whp all cycles of length $\omega(f)$ lie inside L_1 and all cycles of length $o(f)$ outside.

Perhaps surprisingly, the cycle structure in the random planar graph $P(n,m)$ behaves differently. The following result on $P(n,m)$ was proven in [24] by means of random cores and the Pólya urn model as described in Section 3.1. To state Theorem 1, we use the asymptotic notation from Definition 1.

Theorem 1 ([24, 28])**.** *Let $m = n/2 + s$ for $s = s(n) = o(n)$ such that $s^3n^2 \to \infty$. Then whp $c(L_1)$, $g(L_1)$, and $c(R)$ lie in the following ranges.*

	$c(L_1)$	$g(L_1)$	$c(R)$
$G(n,m)$	$\Theta\left(s^2n^{-1}\right)$	$\Theta_p\left(ns^{-1}\right)$	$\Theta_p\left(ns^{-1}\right)$
$P(n,m)$	$O\left(sn^{-1/3}\right)$, $\omega\left(n^{1/3}\right)$	$\Theta_p\left(ns^{-1}\right)$	$\Theta_p\left(n^{1/3}\right)$

This result shows that whp the longest cycle in $P(n,m)$ is still inside the largest component. However, whp the longest cycle outside L_1 is much longer than the shortest cycle contained in L_1, as $ns^{-1} = o\left(n^{1/3}\right)$. Hence, in contrast to $G(n,m)$, there is no threshold function as asked in (Q2) for $P(n,m)$.

4 Random complex part

We recall that the random complex part $Q(C,q)$ is a graph chosen uniformly at random from the class of all complex parts with core C and q vertices (see Definition 2). In Section 2 we have seen that we obtain the complex part of a graph by replacing each vertex in the core by a rooted tree. Thus, we can construct $Q(C,q)$ as follows. We choose uniformly at random a forest F on vertex set $[q]$ with $v(C)$ trees such that the vertices of C, i.e. vertices with labels in $[v(C)]$, lie in different trees. We consider the vertices in $[v(C)]$ as the roots of the trees in F. Then we replace each vertex v in C by the tree in F rooted at v. We note that the only randomness in this construction lies in the choice of F. Therefore, we consider

the random forest F in more detail in Section 4.1. In particular, we obtain a generalised version of Prüfer sequences, which allows to construct F (and therefore also $Q(C,q)$) via a balls–into–bins model. In [25] this approach was the starting point of showing a strong concentration result on the maximum degree in sparse random planar graphs, which we state in Section 4.2.

4.1 Random forests, Prüfer sequences, and balls into bins

For given $n,t \in \mathbb{N}$ we let $\mathcal{F}(n,t)$ be the class of all forests on vertex set $[n]$ having t trees such that all vertices in $[t]$ lie in different trees. We denote by $F(n,t)$ a forest chosen uniformly at random from the class $\mathcal{F}(n,t)$. We call $F(n,t)$ a *random forest with specified roots* or for short just a random forest. The vertices with labels in $[t]$ are called *roots*.

Let $\mathcal{S}(n,t) := [n]^{n-t-1} \times [t]$ be the set of all sequences of length $n-t$ with elements in $[n]$ such that the last element lies in $[t]$. In the following, we describe a bijection between $\mathcal{F}(n,t)$ and $\mathcal{S}(n,t)$ similar to Prüfer sequences for trees (see e.g. [32, 41]). Given a (deterministic) forest $F \in \mathcal{F}(n,t)$ we recursively delete the leaf, i.e. a vertex of degree one, with largest label and note the unique neighbour of it. We call the sequence $(x_1,\dots,x_{n-t})$ of neighbours the *Prüfer sequence* $\psi(F)$ of F. We observe that we never delete a root vertex $r \in [t]$ in F, since r is always the vertex with smallest label in the component containing r. Hence, after $n-t$ deletions we have removed all the vertices from $[n] \setminus [t]$. In particular, the neighbour x_{n-t} of the last removed vertex lies in $[t]$, which shows that $\psi(F)$ is indeed in $\mathcal{S}(n,t)$.

The fact that the removed vertices are exactly those in $[n] \setminus [t]$ implies the following relation between the degree $d_F(v)$ of a vertex $v \in [n]$ and the number of occurrences $\#(v,\psi(F)) := |\{i \in [n-t] \mid x_i = v\}|$ of v in the Prüfer sequence $\psi(F) = (x_1,\dots,x_{n-t})$:

$$d_F(v) = \begin{cases} \#(v,\psi(F)) & \text{if } v \in [t] \\ \#(v,\psi(F)) + 1 & \text{if } v \in [n] \setminus [t]. \end{cases} \tag{16}$$

Using (16) we can reconstruct a forest F if its Prüfer sequence $\psi(F) = (x_1,\dots,x_{n-t})$ is given. Due to (16) the leaf y_1 with largest label in F is determined. Thus, the edge x_1y_1 was the first one which was deleted in the process of building $\psi(F)$. Decreasing the degrees of x_1 and y_1 both by one, we obtain the degrees of the vertices in the forest $F - x_1y_1$. Repeating that argument we can determine all edges of F and therefore F itself. Using this algorithm it is straightforward to show that $\psi : \mathcal{F}(n,t) \to \mathcal{S}(n,t)$ is indeed a bijection (see [25, Theorem 6.1] for details).

We observe that this bijection gives also an alternative proof of Cayley's formula that $|\mathcal{F}(n,t)| = tn^{n-t-1}$ (see e.g. [40]). In the case of trees, i.e. when $t = 1$, the last symbol of $\psi(F) \in \mathcal{S}(n,1)$ is always 1. By ignoring that element we obtain

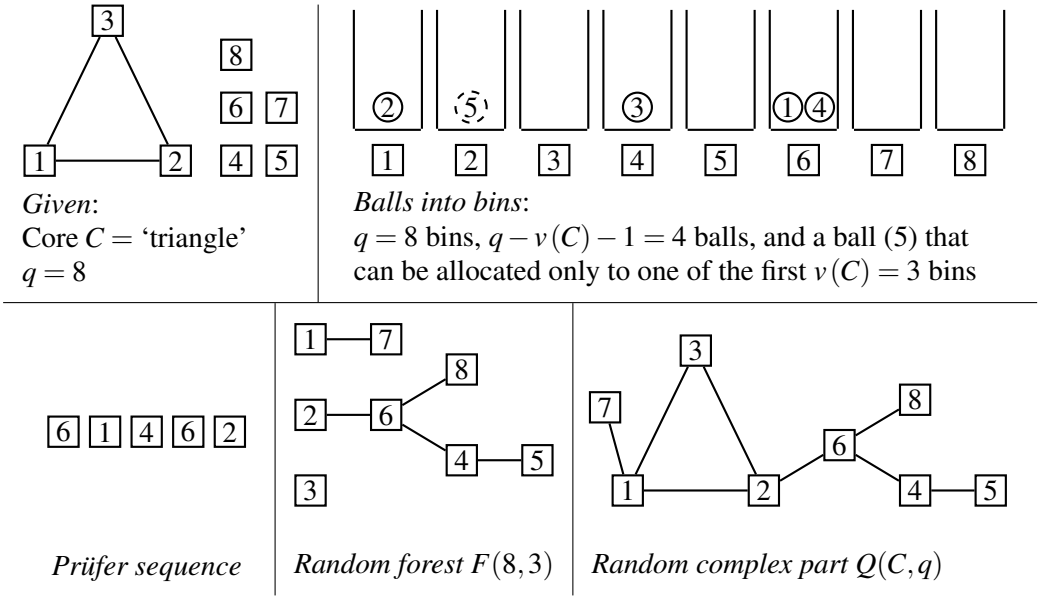

Figure 3: Construction of a random complex part: A balls–into–bins experiment is translated (deterministically) to a Prüfer sequence, a random forest, and finally a random complex part.

the well–known Prüfer sequence of length $n-2$ for trees. We note that in the literature it is more common to delete in each step the leaf with smallest label of a tree. However, in the general case of forests it seems more convenient to consider the leaf with largest label, since then we never delete a root vertex, which makes the analysis much simpler.

Instead of picking a random forest $F = F(n,t)$ directly, we can equivalently choose an element S uniformly at random from $\mathcal{S}(n,t)$ and let F be such that $\psi(F) = S$. To construct the sequence $S = (A_1,\dots,A_{n-t})$, we consider a balls–into–bins model with n bins $\mathcal{B}_1,\dots,\mathcal{B}_n$ and $n-t$ balls $B_1,\dots,B_{n-t}$. For each $i \in [n-t]$ we allocate the ball B_i independently and uniformly at random to one of the bins $\mathcal{B}_1,\dots,\mathcal{B}_n$. We have the additional restriction that the last ball B_{n-t} has to be assigned to one of the first t bins. Then we let A_i be the index of the bin to which the ball B_i is allocated.

To sum up, we can construct a random complex part $Q(C,q)$ by transferring the result of a balls–into–bins experiment deterministically to a Prüfer sequence, a forest, and a complex part (see also Figure 3). Thus, results on the balls–into–bins model (see e.g. [18, 22, 36, 38]) translate to properties of $Q(C,q)$. For example, in [25] the maximum number of balls in a bin was studied to obtain a strong concentration statement on the maximum degree in $P(n,m)$, which we consider in Section 4.2 in more detail.

4.2 Maximum degree in random graphs

Bollobás [2, 3] studied the degrees of vertices in the Erdős–Rényi random graph $G(n,m)$ (see also [5, Chapter 3] for an overview). Amongst other results, he showed that whp the maximum degree $\Delta(G(n,m))$ takes at most two different values, provided that m is not too 'large'.

Theorem 2 ([3]). *Let $m = m(n) = o(n \log n)$ and $G = G(n,m)$ be the Erdős–Rényi random graph. Then there exists a $D = D(n) \in \mathbb{N}$ such that whp $\Delta(G) \in \{D, D+1\}$.*

In [25] a comparable statement for random planar graphs was proven.

Theorem 3 ([25]). *Let $m = m(n)$ be such that $\limsup_{n\to\infty} m/n < 1$ and $P = P(n,m)$ be the random planar graph. Then there exists a $D = D(n) \in \mathbb{N}$ such that whp $\Delta(P) \in \{D, D+1\}$.*

We note that a planar graph on n vertices has at most $3n-6$ edges, while a general graph can contain up to $\binom{n}{2}$ edges. Thus, it seems natural that the upper bound on the number of edges in Theorem 3 is smaller than that in Theorem 2.

In [25] the following strategy was used to obtain Theorem 3. First it is shown that whp the maximum load of a balls–into–bins model, i.e. the maximum number of balls in a bin, is concentrated at two values. Using the construction of the random complex part $Q(C,q)$ as in Figure 3, this result translates to a two–point concentration of $\Delta(Q(C,q))$. Then the core–kernel approach is applied to obtain that whp $\Delta(Q(P))$ takes one of two values, where we recall that $Q(P)$ is the complex part of the random planar graph $P = P(n,m)$. Finally, Theorem 3 follows by the fact that whp also the maximum degree in $P \setminus Q(P)$, i.e. P without the complex part $Q(P)$, is concentrated at two values.

We note that more general results than Theorem 3 are proven in [25]. For example, it is shown that whp $\Delta(P) = (1+o(1)) \log n / \log\log n$ if $\liminf_{n\to\infty} m/n > 0$ and $\limsup_{n\to\infty} m/n < 1$. Moreover, there are concentration results on the maximum degree in the largest component of P provided.

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Stochastic differential equations with irregular coefficients: mind the gap!

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Numerical methods for stochastic differential equations with non-globally Lipschitz coefficients are currently studied intensively. This article gives an overview of our work for the case that the drift coefficient is potentially discontinuous complemented by other important results in this area. To make the topic accessible to a broad audience, we begin with a heuristic on SDEs and a motivation.

1 SDEs in a nutshell

A dynamical system is usually described as a solution to a deterministic (ordinary) differential equation (ODE)

$$x\colon [0,T] \to \mathbb{R}, \qquad \frac{d}{dt}x(t) = \mu(x(t)), \qquad x(0) = 1.$$

If we allow for random disturbance W scaled by a coefficient σ in the dynamics, this becomes

$$\frac{d}{dt}X(t) = \mu(X(t)) + \sigma(X(t))\frac{d}{dt}W_t, \qquad X(0) = 1.$$

An issue here is that we cannot formally differentiate the (Wiener) noise process W. Hence, in stochastic analysis the *stochastic differential equation (SDE)* is only a formal notation for the corresponding integral equation, taking the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \qquad X(0) = 1.$$

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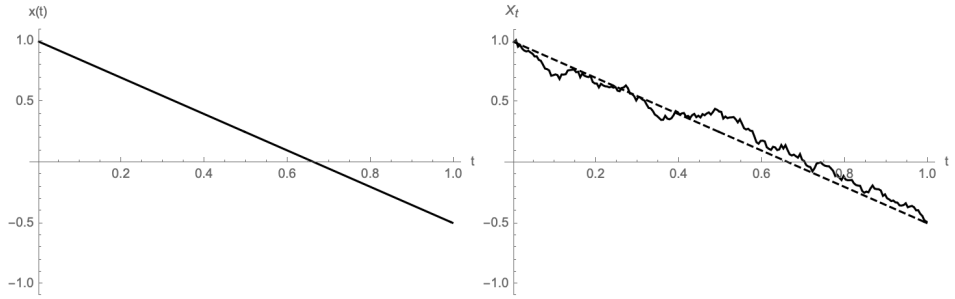
A solution to a stochastic differential equation is a stochastic process[1]. We illustrate the difference between the deterministic and the stochastic case for $\mu \equiv -1.5$ and constant σ.



The graph on the left-hand side shows the solution to the ODE, the graph on the right-hand side displays the solution to the ODE and a solution path[2] of the SDE. We observe that the solution path "zick-zacks" randomly around the trend.

2 Numerical methods for SDEs

Our objects of study are SDEs of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = \xi, \tag{17}$$

where $\xi \in \mathbb{R}^d$, $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$, and W is a d-dimensional Brownian motion. The coefficient μ is called *drift*, σ is called *diffusion*.

Our example in Section 1 was solved explicitly. However, as in the case of ODEs, for solving most of the SDEs we have to resort to numerics. The simplest numerical method is the *Euler–Maruyama method*. This approximation scheme is very similar to the Euler method for ODEs, only the noise term is added. We denote the Euler–Maruyama approximation of the solution to SDE (17) by X^δ; δ is the maximal step size.

Algorithm 1. *The Euler–Maruyama method is defined as follows.*

- *Choose a time grid* $0 = t_0 < t_1 < \ldots < t_n = T$ *and set* $\delta := \max\{t_{k+1} - t_k\}$.
- *Start at time 0:* $X^\delta_{t_0} = \xi$
- *Choose* $\Delta_{k+1} \sim \mathcal{N}(0, t_{k+1} - t_k)$ *and set*

$$X^\delta_{t_{k+1}} = X^\delta_{t_k} + \mu(X^\delta_{t_k}) \cdot (t_{k+1} - t_k) + \sigma(X^\delta_{t_k}) \cdot \Delta_{k+1}.$$

[1] A *stochastic process* is a function $X\colon [0,T] \times \Omega \to \mathbb{R}^d$ where for all $t \in [0,T]$, $X(\cdot,t)$ is measurable. Thereby $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

[2] For every $\omega \in \Omega$, $X(\omega,\cdot)$ is called a *path* of the stochastic process X.

As simple and explicit schemes are desirable, we stick to the Euler–Maruyama method here. The question is now: does this method work? The following theorem by Maruyama gives a first answer.

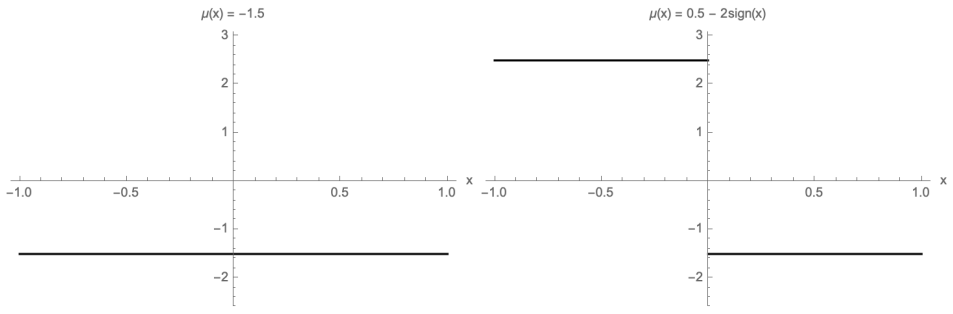
Theorem 1. *If μ and σ are globally Lipschitz continuous, then the Euler–Maruyama method*[3] *has L^2-convergence order $1/2$, that is*

$$\left(\mathbb{E}\left[\|X_t - X_t^\delta\|^2\right]\right)^{1/2} \leq c \cdot \delta^{1/2}.$$

SDEs with irregular coefficients

Now we may ask ourselves: are we happy with this? If the coefficients that appear in your application satisfy global Lipschitz conditions, then your answer might be yes! However in many applications the coefficients are not globally Lipschitz. In that case we speak of *irregular coefficients*. Here we are particularly concerned with SDEs with discontinuous drift. This situation frequently appears, for example, in optimal control. Think of a light switch: the light can either be on or off, influencing the drift of some underlying stochastic system. Similar discontinuities appear in energy market models, or in insurance mathematics and mathematical finance, where switching on the light is for example replaced by paying dividends to shareholders. This motivates studying SDEs with irregular coefficients from an application point of view. But – to be honest – the most important motivation for many of us is mathematical curiosity. When studying SDEs with discontinuous drift, the first question that might come to your mind is

Do discontinuities matter? First, let us think of the ODE we solved in Section 1, where we chose $\mu \equiv -1.5$ (plotted on the left-hand side below). Then, introduce a discontinuity in the point 0 (plotted on the right-hand side below).

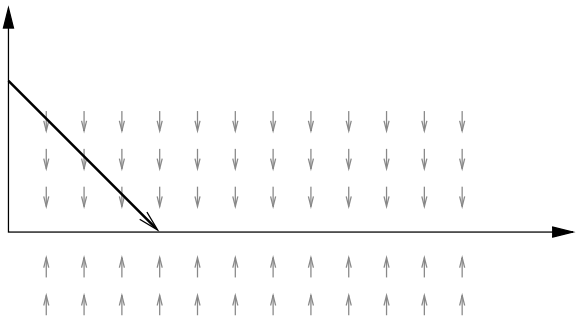


[3] In this paper convergence results are always given for the time-continuous versions of the schemes.

The ODE corresponding to the drift on the left-hand side is explicitly solvable as we saw in Section 1. The ODE corresponding to the drift on the right-hand side,

$$\frac{d}{dt}x(t) = 0.5 - 2\,\mathrm{sign}(x(t)), \qquad x_0 = 1, \tag{18}$$

does not admit a solution. Heuristically speaking, the reason for this is that whenever the trajectory hits zero, it is simultaneously pushed to the positive and to the negative and cannot proceed. This is illustrated in following figure[4].

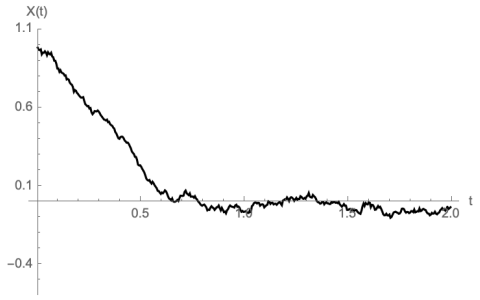


This answers our question to the extent, that discontinuities do indeed matter for the question of existence of solutions of ODEs. It also shows that one has to be careful with applying numerical solvers. While the Euler method would actually lead to a result for ODE (18), this result would be nonsensical.

Now let us add noise again:

$$dX_t = (0.5 - 2\,\mathrm{sign}(X_t))dt + \sigma dW_t, \qquad X_0 = 1.$$

In this case a solution exists. In fact, the Brownian noise has a regularising effect. The noise constantly pushes the trajectory away from the point of discontinuity 0, see below.



Existence and uniqueness of solutions to SDEs with discontinuous drift is true under very general conditions and has been studied in various setups, see, e.g.,

[4]The figure was kindly provided by Gunther Leobacher.

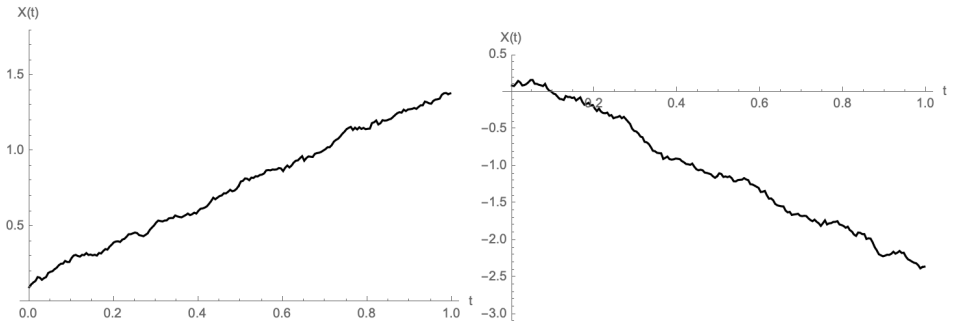
[43, 36, 37, 38, 14, 42, 20, 34, 15, 39, 16, 40, 32, 21, 33, 35].

Now that existence and uniqueness of solutions is settled, let us turn to numerics. Can irregularities in the coefficients matter when it comes to numerics? This leads us to a number of papers published between 2015 and 2018 where the authors construct explicit examples of SDEs with smooth and bounded but non-Lipschitz coefficients, for which they show that the Euler–Maruyama method converges, but also that any numerical method will converge arbitrarily slowly, see [8, 12, 22, 41, 5, 23]. While these are academic examples, [10] shows a similar result for the well-known Heston model from mathematical finance for some parameter configurations. This model is used in practice; the parameters are determined by the market, so we cannot choose them freely; for some cases simulations in this model will not converge in reasonable time, which would however be essential for pricing financial derivatives. All these examples show impressively that irregularities can indeed matter for numerics.

However, in the above examples the problems are not about discontinuities. In [6] the authors performed a numerical study on SDEs with discontinuous drift coefficient. They observe a strange effect: the convergence order seems to be too low in the case where the drift points away from the discontinuity. A similar effect was observed by [32] in presence of jump noise. A possible reason for this can be found in rare-event simulation; too few paths "find" the discontinuity. For the following SDE with outward pointing drift

$$dX_t = -(0.5 - 2\,\mathrm{sign}(X_t))dt + dW_t, \qquad X_0 = 0.1$$

the left-hand side plot shows a typical path; the right-hand side plot shows a path that hits the discontinuity. This happens so rarely that the author managed to produce the right-hand side path only by manipulating the data.



This shows that in case of a discontinuous drift we have to be even more careful with interpreting numerical results; theoretical results are particularly important.

3 SDEs with piecewise Lipschitz drift

A natural assumption that allows for discontinuities in the drift and that is satisfied in many of the above mentioned applications is piecewise Lipschitz continuity.

Definition 1 ([15, Definition 2.1])**.** *A function $f\colon \mathbb{R} \to \mathbb{R}$ is piecewise Lipschitz, if there exist $m \in \mathbb{N}$ and $\zeta_1, \dots, \zeta_m \in \mathbb{R}$ with $\zeta_1 < \dots < \zeta_m$ such that f is Lipschitz on each of the intervals $(-\infty, \zeta_1)$, (ζ_1, ζ_2), $\dots$, (ζ_{m-1}, ζ_m), (ζ_m, ∞).*

The points ζ_k in Definition 1 are possible points of discontinuity of the piecewise Lipschitz function f. A multidimensional extension of this definition is available, but requires notions from differential geometry, see [16].

One idea for solving SDEs with piecewise Lipschitz drift coefficient is to use a transformation-based method.

Algorithm 2 (Leobacher and Szölgyenyi [15, 16])**.** *The transformation method works as follows:*

- *Construct a transform G depending on the coefficients of the SDE such that the SDE for $G(X)$ has Lipschitz coefficients;*
- *compute its inverse G^{-1};*
- *define $Z = G(X)$ and calculate the SDE for Z by Itô's formula;*
- *apply the Euler–Maruyama scheme to compute the approximate solution Z^δ to this SDE;*
- *define the approximation $\bar{X}$ of the solution to the original SDE by $\bar{X} = G^{-1}(Z^\delta)$.*

Theorem 2 (Leobacher and Szölgyenyi [15, 16])**.** *If μ is piecewise Lipschitz, σ is Lipschitz, and $\sigma(\text{points of discontinuity}) > 0$, then the transformation method has strong convergence order $1/2$.*

The multidimensional version of Theorem 2 additionally requires conditions on the hypersurface of discontinuity, see [16, 19]. An overview can be found in [17].

Now we know a numerical method for SDEs with discontinuous drift of which we know that it converges at the highest rate that can be expected. However, there are two shortcomings of the method. First, in the multidimensional case G has to be inverted numerically, which is costly. Second, we have to know the points of discontinuity for the construction of G. Numerically calculating those points, e.g., in the case of applications in optimal control, where the discontinuity is imposed by the control, is also costly. So actually we prefer a simple and explicit method.

What about the Euler–Maruyama method? We can make use of the transformation idea from above to split the error of the Euler–Maruyama approximation X^δ into the approximation error of the Euler–Maruyama approximation of $Z = G(X)$, that is $\mathbb{E}[\|Z_t - Z_t^\delta\|^2]$, and the difference between the approximation error of the transformed equation and the transformation of the approximation error of the Euler–Maruyama approximation of X, that is $\mathbb{E}[\|Z_t^\delta - G(X_t^\delta)\|^2]$:

$$\begin{aligned}\mathbb{E}\left[\|X_t - X_t^\delta\|^2\right] &= \mathbb{E}\left[\|G^{-1}(Z_t) - G^{-1}(G(X_t^\delta))\|^2\right] \\ &\leq 2(L_{G^{-1}})^2\mathbb{E}\left[\|Z_t - Z_t^\delta\|^2\right] + 2(L_{G^{-1}})^2\mathbb{E}\left[\|Z_t^\delta - G(X_t^\delta)\|^2\right].\end{aligned}$$

The hard part here is estimating the second error term. We obtain the following result.

Theorem 3 (Leobacher and Szölgyenyi [18])**.** *Under the same assumptions as in Theorem 2, the Euler–Maruyama method has essentially strong convergence order* $1/4$.

So on the one hand we have the transformation method that convergences quickly, but is not simple to implement and potentially costly, and we have the simple and explicit Euler–Maruyama method of which up to now we only have convergence order $1/4$.

Can we do better by incorporating more knowledge about the points of discontinuity? The idea is, instead of using a deterministic grid, to choose the step size τ_k adaptively in dependence of the current position of the approximation $\hat{X}$:

$$\tau_0 = 0, \qquad \tau_{k+1} = \tau_k + h(\hat{X}_{\tau_k}, \delta)$$

for a step size function h that decreases whenever $\hat{X}_{\tau_k}$ gets closer to the discontinuity of the drift and where δ is again the maximal step size. For the exact choice of h, see [28]. The resulting adaptive Euler–Maruyama scheme looks quite similar to the classical one; only the step sizes are now random variables as well:

$$\hat{X}_0 = \xi, \qquad \hat{X}_{\tau_{k+1}} = \hat{X}_{\tau_k} + \mu(\hat{X}_{\tau_k})(\tau_{k+1} - \tau_k) + \sigma(\hat{X}_{\tau_k})(W_{\tau_{k+1}} - W_{\tau_k}).$$

When studying adaptive methods, it is not sufficient to estimate the rate of convergence, but we also have to estimate the computational cost. For the classical Euler–Maruyama scheme the computational cost is proportional to the (deterministic) number of steps, which is itself proportional to $1/\delta$. A good rate of convergence does not help, if the number of steps $N(h)$, which depends via h on $\hat{X}$ and hence is random, explodes.

Theorem 4 (Neuenkirch et al. [28])**.** *Under the same assumptions as in Theorem 2, the rate of convergence of the adaptive Euler–Maruyama scheme is essentially* $1/2$*. The expected computational complexity is essentially proportional to* $1/\delta$*, that is for all* $\varepsilon \in (0,1)$ *there exists* $c \in (0,\infty)$ *such that*

$$\mathbb{E}[N(h)] \leq c \cdot \delta^{-1+\varepsilon}.$$

So we have found a numerical method that converges at essentially the highest expectable rate at essentially the same computational cost. This is the best result for the multidimensional case we know so far. In the scalar case however there has been a significant improvement to Theorem 3. The setup is the same, but in the analysis of the error term $\mathbb{E}[\|Z_t^\delta - G(X_t^\delta)\|^2]$, in particular in a critical point of the analysis, a better estimate has been achieved, yielding the optimal convergence order.

Theorem 5 (Müller-Gronbach and Yaroslavtseva [25])**.** *Under the same assumptions as in Theorem 2 and* $d = 1$*, the Euler–Maruyama method has strong convergence order* $1/2$*.*

There is also a very recent result on higher order methods for SDEs with discontinuous drift. The higher order method that has been studied is a transformation-based Milstein scheme and is obtained by replacing the Euler–Maruyama scheme in Algorithm 2 by the Milstein scheme.

Theorem 6 (Müller-Gronbach and Yaroslavtseva [24])**.** *The transformation-based Milstein scheme has strong convergence order* $3/4$*.*[5]

In the regular case the convergence order of the Milstein scheme is 1. So this is the convergence order we would expect also in our case. In Section 5 below we will come back to this issue.

In the case where our SDE is scalar and additionally contains Poisson jumps, that is

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + \rho(X_t)dN_t, \qquad X_0 = \xi,$$

the optimal convergence order or the Euler–Maruyama method has been proven:

Theorem 7 (Przybyłowicz and Szölgyenyi [32])**.** *In the case of presence of Poisson jumps and* $d = 1$ *the Euler–Maruyama scheme has strong convergence order* $1/2$*.*[6]

[5] For the assumptions see [24].

[6] For the assumptions, in particular for those on the jump coefficient ρ, see [32].

Additional literature

Further contributions on the topic are [7, 9, 29, 30, 31]. For SDEs with one-sided Lipschitz drift, [7] proves an almost sure convergence result. For discontinuous but monotone drift coefficient and additive noise (i.e. $\sigma \equiv 1$), [9] provides strong convergence of the Euler–Maruyama scheme. In the multidimensional case, for one-sided Lipschitz drift that is an appropriate limit of smooth functions, [29] proves L^2-convergence order $1/4$. In [30] they extend this result to not necessarily one-sided Lipschitz drift functions for scalar SDEs, and in [31] they also allow for certain discontinuous diffusion coefficients. The last three results require a uniformly non-degenerate diffusion coefficient, which is not required in the results presented above. For scalar SDEs with additive noise the most general result is [1], where they prove an L^2-order of essentially $1/2$ even for drift coefficients which are only bounded and integrable.

Weak convergence results have been obtained in [13, 4]. In [2, 3] the authors present an exact simulation algorithm for scalar SDEs with drift that is discontinuous in one point, but differentiable everywhere else.

4 Dependence of the convergence order on the regularity of the drift

Now that we know quite a number of results for SDEs with discontinuous drift, let us take a step back and ask a more general question: how does the convergence order of the Euler–Maruyama method depend on the regularity of the drift coefficient? For this we consider a more simple setup, that is we study scalar SDEs with additive noise:

$$dX_t = \mu(X_t)dt + dW_t, \qquad X_0 = \xi,$$

where $\mu\colon \mathbb{R} \to \mathbb{R}$ is the (not necessarily continuous) drift coefficient.

If μ is bounded and measurable, [43] ensures existence and uniqueness of a solution. So we do not have to worry about existence and uniqueness and can directly proceed to numerics.

In [27] we provide a novel framework for the error analysis: we decompose the error into a discretisation error and an error coming from approximating a quadrature problem for Brownian motion.

Theorem 8 (Neuenkirch and Szölgyenyi [27, Theorem 2.4])**.** *Assume that μ is bounded and can be decomposed into a regular part $a \in C_b^2(\mathbb{R},\mathbb{R})$ and an irregular part $b \in L^1(\mathbb{R},\mathbb{R})$, that is $\mu = a + b$. Then for all $\varepsilon \in (0,1)$, there exists $c \in (0,\infty)$ such that*

$$\mathbb{E}\left[|X_T - X_T^\delta|^2\right] \leq c \cdot \left(\delta^2 + \mathcal{W}^{1-\varepsilon}\right),$$

where

$$\mathcal{W} = \mathbb{E}\left[\left|\int_0^T G'(W_s+\xi)\left(b(W_s+\xi)-b(W_{\underline{s}}+\xi)\right)ds\right|^2\right],$$

and where G is a Zvonkin-type transform (see [43, 27]) for the irregular part of the drift.

It remains to analyse the error from approximating the quadrature problem $\mathcal{W}$. For this we choose an appropriate function space for b where the regularity of b is determined by a parameter κ; the higher κ the more regular is b. Under the additional assumption that b lies in this function space, we get the following overall approximation result, where the error depends, as desired, on the regularity of the drift.

Theorem 9 (Neuenkirch and Szölgyenyi [27, Corollary 3.9])**.** *Under the same assumptions as in Theorem 8 and under the additional assumption that there exists* $\kappa \in (0,1)$ *such that*

$$|b|_\kappa := \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|b(x)-b(y)|^2}{|x-y|^{2\kappa+1}}dxdy\right)^{1/2} < \infty,$$

i.e. b belongs to the fractional Sobolev-Slobodeckij space of order κ*, we have for all* $\varepsilon \in (0,1)$ *that there exists a constant* $c \in (0,\infty)$ *such that the Euler–Maruyama scheme satisfies*

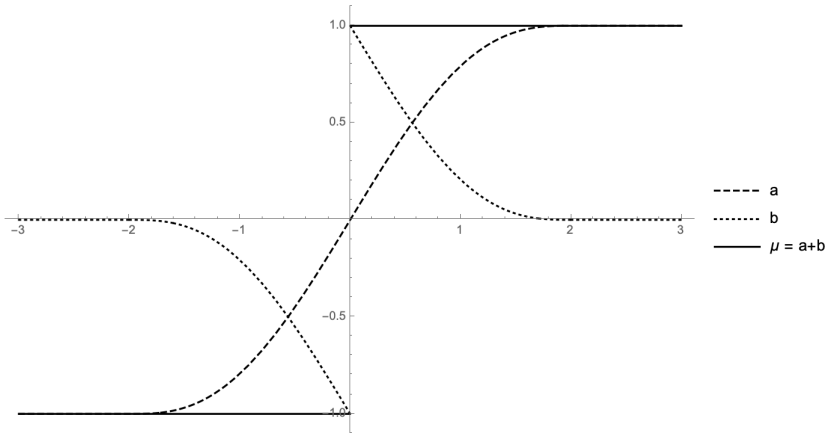
$$\left(\mathbb{E}\left[\|X_t - X_t^\delta\|^2\right]\right)^{1/2} \leq c \cdot \delta^{(1+\kappa)/2-\varepsilon},$$

i.e. the convergence order is essentially $(1+\kappa)/2$.

Note that in the case $a \equiv 0$, by [1] the above result also holds for $\kappa = 0$.

But which kind of functions belong to a fractional Sobolev-Slobodeckij space of order κ? This finally brings us back to our example from Section 2.

Example 1 (Neuenkirch and Szölgyenyi [27])**.** *Let* $\mu(x) = \operatorname{sign}(x)$*. A decomposition* $\mu(x) = a(x) + b(x)$ *which satisfies the assumptions of Theorem 9 for all* $\kappa < 1/2$ *is*



Hence, the convergence order of the Euler–Maruyama scheme for the SDE with $\mu(x) = \mathrm{sign}(x)$ *is essentially* $3/4$.

5 What is behind the rate $3/4$?

Note that for SDEs with additive noise and Lipschitz drift the convergence order of the Euler–Maruyama scheme is 1. In Example 1 we only obtain rate $3/4$. Recall that we made a similar observation in Section 3: under classical assumptions the Milstein scheme has convergence order 1, but in the case of a discontinuous drift [24] (here Theorem 6) show convergence order $3/4$ of a transformation-based Milstein scheme. Were our estimates too coarse, or is there a structural difference between the convergence orders in the classical case and in the case of a discontinuous drift? This question is answered by the following very recent result.

Theorem 10 (Müller-Gronbach and Yaroslavtseva [26])**.** *For scalar SDEs with additive noise the convergence order of any numerical method on a finite deterministic grid is at most* $3/4$.[7]

Previous work on lower error bounds for scalar SDEs with discontinuous drift can be found in [11].

The last theorem shows that there is indeed a structural difference between SDEs with Lipschitz drift and SDEs with discontinuous drift. So yes indeed, discontinuities do matter!

Acknowledgement

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[7] For the assumptions see [26].

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Buchbesprechungen

M. V. Markin: Elementary Operator Theory (P. GRABNER) 57
M. Hils, F. Loeser: A First Journey through Logic (C. FREI) 58

M. V. Markin: Elementary Operator Theory. De Gruyter, 2020, 410 S. ISBN 978-3-11-060096-4 P/b EUR 64,95.

Das vorgestellte Buch leistet genau das, was es im Titel verspricht: eine elementare Einführung in die Theorie linearer Operatoren. Tatsächlich setzt es wenig Vorkenntnisse voraus und liefert diese auf den ersten etwa 100 Seiten: mengentheoretische Grundlagen, Grundtatsachen der linearen Algebra, metrische Räume und deren Topologie. Darauf aufbauend, wird die Theorie der Banach-Räume und deren linearer Operatoren gut verständlich aufbereitet. Daran anschließend wird die Spektraltheorie zuerst allgemein für Operatoren auf Banach-Räumen und dann auf Hilbert-Räumen entwickelt. Ein Appendix über das Auswahlaxiom und dessen Äquivalente dazu rundet das Buch ab. Durch die vielen in den Text eingestreuten Übungsaufgaben und die Problemsammlung am Ende jeden Kapitels ist das Buch bestens zum Selbststudium als auch als Grundlage für einen Kurs geeignet.

P. Grabner (Graz)

M. Hils, F. Loeser: A First Journey through Logic. (Student Mathematical Library Vol. 89.) American Mathematical Society, 2019, 185 S. ISBN 978-1-4704-5272-8 P/b $ 55.

Das Werk bietet auf 178 Seiten eine sehr kompakte Einführung in die verschiedenen Bereiche der mathematischen Logik. Hierbei versuchen die Autoren nicht, die Logik als Fundament der Mathematik aufzubauen, sondern präsentieren sie als gleichwertig mit anderen Teilgebieten. Insbesondere werden von Beginn an Konzepte wie die natürlichen Zahlen und naive Mengenlehre verwendet.

Das erste Kapitel führt Ordinal- und Kardinalzahlen ein. Darauf folgt ein Kapitel über Prädikatenlogik erster Ordnung, das mit Gödels Vollständigkeitssatz endet. Kapitel 3 behandelt die Modelltheorie. Nach einer Einführung wesentlicher Resultate wie des Kompaktheitssatzes und des Satzes von Löwenheim-Skolem wird insbesondere die Theorie ACF der algebraisch abgeschlossenen Körper behandelt, inklusive einer Version des Lefschetz-Prinzips aus der algebraischen Geometrie. Kapitel 4 bietet eine Einführung in die Berechenbarkeit; hier wird gezeigt, dass die durch eine Turingmaschine berechenbaren Funktionen genau die rekursiven Funktionen sind. Kapitel 5 behandelt die Peano-Arithmetik und ihre Modelle, insbesondere die Gödelschen Unvollständigkeitssätze. Das sechste und letzte Kapitel führt zurück zur Mengenlehre mit einer Behandlung der ZFC-Axiome inklusive diverser Unabhängigkeitsresultate.

Das Werk enthält viele umfangreiche Übungsaufgaben, die manchmal später vorausgesetzt werden. Die Beweise sind präzise, aber sehr knapp formuliert und setzen daher eine gewisse mathematische Reife voraus. Für einen ersten Kontakt mit der Logik erscheint mir die Präsentation etwas zu knapp. Motivierende Diskussionen werden auf ein Minimum beschränkt, und die Aussagenlogik wird beispielsweise in weniger als einer Seite abgehandelt. Für Mathematiker aus anderen Gebieten, die einen Überblick über grundlegende Themen der Logik erhalten wollen, ist das Werk sehr gut geeignet.

C. Frei (Graz)

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