Internationale Mathematische Nachrichten

International Mathematical News

Nouvelles Mathématiques Internationales

Die IMN wurden 1947 von R. Inzinger als „Nachrichten der Mathematischen Gesellschaft in Wien“ gegründet. 1952 wurde die Zeitschrift in „Internationale Mathematische Nachrichten“ umbenannt und war bis 1971 offizielles Publikationsorgan der „Internationalen Mathematischen Union“.

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Herausgeber:

Österreichische Mathematische Gesellschaft, Wiedner Hauptstraße 8–10/104, A-1040 Wien. email *imn@oemg.ac.at*, *http://www.oemg.ac.at/*

Bezug:

Die IMN erscheinen dreimal jährlich und werden von den Mitgliedern der Österreichischen Mathematischen Gesellschaft bezogen.
Jahresbeitrag: € 35,–

Bankverbindung: IBAN AT83-1200-0229-1038-9200, bei der Bank Austria-Creditanstalt (BIC-Code BKAUATWW).

Eigentümer, Herausgeber und Verleger: Österr. Math. Gesellschaft. Satz: Österr. Math. Gesellschaft. Druck: Weinitzendruck, 8044 Weinitzen.

ISSN 0020-7926

Österreichische Mathematische Gesellschaft

Gegründet 1903
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Vorstand:

Beirat:

Vorsitzende der Sektionen und ständigen Kommissionen:

Diese gehören statutengemäß dem Beirat an.

Mitgliedsbeitrag:

Jahresbeitrag: € 35,–

Bankverbindung: IBAN AT8312000 22910389200 bei der Bank Austria–Creditanstalt (BKAUATWW).

Internationale Mathematische Nachrichten

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Nouvelles Mathématiques Internationales

Nr. 231 (70. Jahrgang) **April 2016**

Inhalt

Die Titelseite zeigt eine Triangulierung eines Kreisrings und steht symbolisch für das Gebiet der Cluster-Algebren, das hier nicht beschrieben werden soll und das in diesem Heft der IMN prominent in Verbindung zur Musik vorkommt. Für die Definition von Cluster-Algebren und die damit zusammenhängenden Fries-Muster und Triangulierungen von Flächen sei auf S. Fomin, M. Shapiro, and D. Thurston, *Cluster algebras and triangulated surfaces, I. Cluster complexes*. Acta Math. 201, 83–46, 2008, verwiesen.

Internat. Math. Nachrichten
Nr. 231 (2016), 1–14

Mathematics and Arts: Towards a balance between artistic intuition and mathematical complexity

Karin Baur, Klemens Fellner
University of Graz

1 Introduction

An engineer, a physicist and a mathematician travel on a train through Scotland and watch the passing landscape. Suddenly, the engineer exclaims: "Hey, there are black sheep in Scotland!", upon which the physicist corrects: "Actually, we can only say that there is at least one black sheep in Scotland." The mathematician says: "Precisely speaking, there is at least one sheep with one black side in Scotland".

Recalling this first-year's joke, is it a funny thing to talk of one-sided sheep?

Let's ask a different question: What would happen if there were also an *artist* present? What would an artist associate with a one-sided sheep? And what might happen if the mathematician and the artist were to join teams?

The research project entitling this report brings together two mathematicians and two artists of separate fields: Karin Baur (Algebra), Gerhard Eckel (Computer-music, Sound-installations), Klemens Fellner (Partial Differential Equations and Applications), and Tamara Friebel (Composition, Architecture).

Mathematics and the arts have been linked by common roots since the dawn of human civilisation. They are spoken of as manifestations of inner truths of the world. It is easy to name a few highlights where the expressions of mathematics

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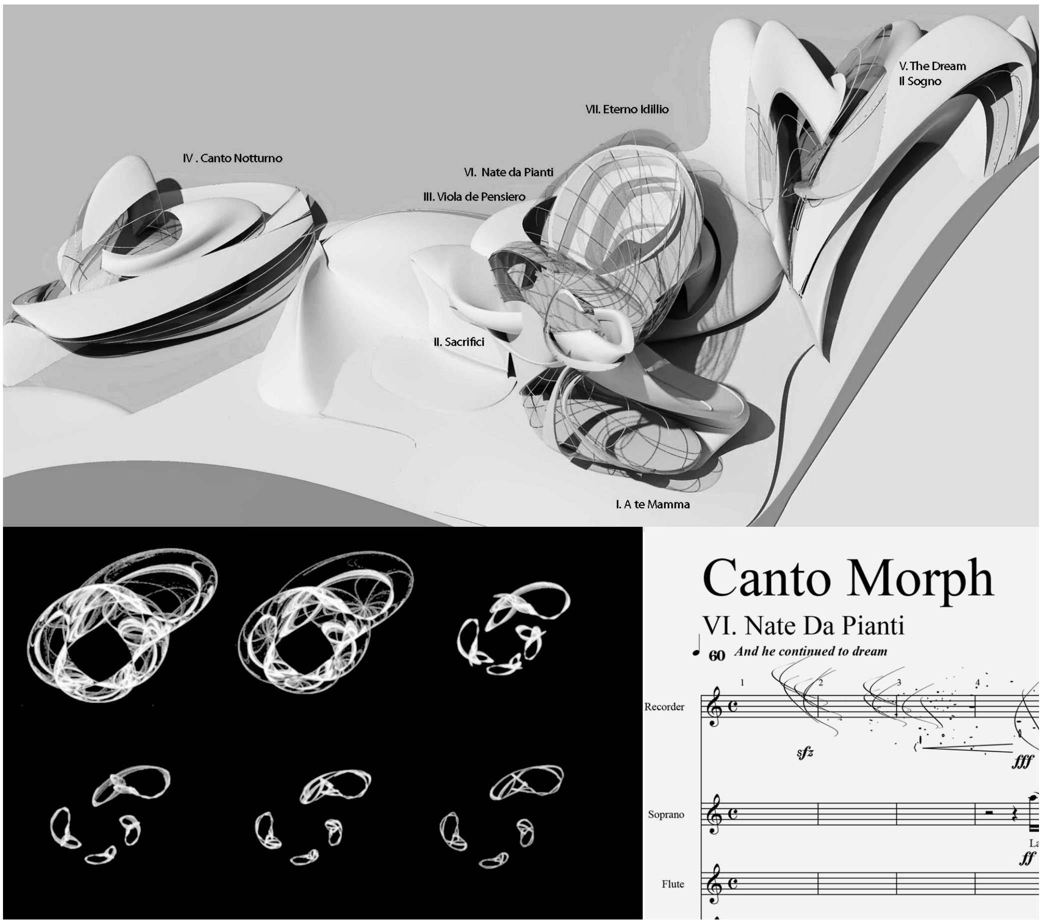


Figure 1: A set of strange attractors and its musical and architectonic derivatives.

and the arts touched one another throughout history: The Vedic scriptures and their intrinsic realisations of music and architecture, the Greek philosophers and the harmonic world of Pythagoras, the Renaissance with da Vinci's universality and Palladio's architecture.

In the last decades, however, it seems that mathematical structures and dynamical systems have increasingly influenced the arts.

The architect *Le Corbusier* used proportional sequences constructed from the golden ratio and human body proportions to create his outstanding buildings. The composer *Xenakis* (in his early years a co-worker of Le Corbusier) transferred the interest in proportional systems and mathematical/graphical structures into the field of music.

The contemporary designs of architects like *Zaha Hadid* are based on curved geometric forms often discovered by time-costly experimenting with parametric design software like *GenerativeComponents*. In *aleatoric music*, composers turned to stochastic processes in an attempt to overcome the constraints of twelve-tone music and serialism.

The composer *Ligeti*, on the other hand, used a more metaphorical, transformative approach of scientific research in his concept of musical "permeability", which describes the freedom of choice of intervals of a musical structure.

To succeed artistically when utilising mathematical structures or dynamical systems, a crucial link seems to be the individual intuition of the artist.

Le Corbusier wrote eloquently about the construction of his *modulor* sequences, yet failed to convey how a proportional system yields great architecture. In fact, he remained the only leading architect using the modulor, by which means he intended to revolutionise architecture for ever.

Ligeti arrives at a new micropolyphonic method of composition where the polyphony cannot be actively heard by the listener, but remains underwater, hidden from the listener. He thus uncovered novel and fresh methods of listening and organising sound.

Tamara Friebel's work complex *Canto Morph* is a contemporary example of how mathematics can inspire intuitive artistic processes across the fields of architecture and composition. Based on proportional relations found within a strange attractor set, Friebel designed the Canto Morph pavilions, a performance space specifically designed to host the opera "Canto Morph" also composed by Friebel (see Fig. 1). The artistic and intuitive use of mathematical systems was researched by Friebel in the context of composition [10] as well as architecture [11, 12].

In the context of the artistic research project "The Choreography of Sound" [4] Gerhard Eckel used dynamical systems to create movement paths for the spatialisation of sound objects in realtime.

A major disadvantage in using strange attractor sets in any creative process is the lack of control: The proportions of a strange attractor are as uncontrollable as the evolution process of a chaotic system is unpredictable.

2 Aims and communication aspects of the project

Our project aims to *research and explore mathematical structures and dynamical systems*, which offer *control points* for non-mathematicians (like artists, composers or architects) *and inspire an intuitive dialogue between artistic creation and mathematical complexity*.

To illustrate an example of such a mathematical system (which is already well used in electroacoustic music) one can imagine a single leader surrounded by a self-organising swarm of individuals, whose behaviour is the result from following the leader and from interacting with every other individual of the swarm: While the motion of the leader can be easily controlled by a sound artist, the collective flocking of the swarm around the leader constitutes a highly complex dynamical behaviour, which has been used to spatialise music in multi-speaker systems in a controllable (via the leader) yet complexly self-organising way.

Note that the above paragraph does not only describe an example of a suitable mathematical system; it also does so by using *mathematical diction*. The above paragraph describes the leader-swarm behaviour from the viewpoint of the mathematician observing sheep in Scotland (even without mentioning gradient flows on metric spaces, which form the mathematical basis of so many swarming models). The viewpoint of any artist will certainly be quite different.

Therefore, an *intrinsic challenge* of the project is to open communication pathways between mathematical ways of thinking and talking and artistic ways of thinking and speaking. The communication between the project team is of crucial importance; in particular because it is explicitly not a goal of the project to build “design machines” or to derive “composition algorithms”!

On the contrary, our project aims to cultivate a dialogue between mathematicians and artists on equal terms, as a collaborative model: A dialogue, which allows artists to become inspired by mathematics, which they would not have otherwise encountered. A dialogue, which allows mathematicians to approach mathematics, for instance, with the naivety of somebody from outside the field or with a certain kind of artistic liberty.

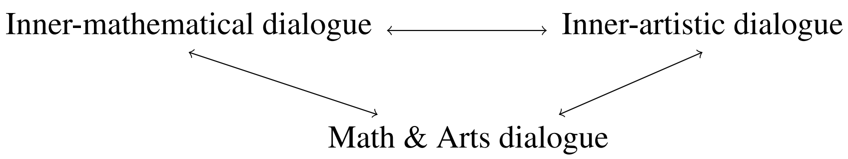
Referring back to the first paragraph of this section, this implies that the question of control points to a complex mathematical system is first and foremost an artistic decision, not a mathematical one. Also, this artistic decision can only be made on the basis of communication between mathematicians and artists.

As a consequence, the meetings of the project team are very rich. Communication and cross-wise inspiration occur on many different levels and these experiences have turned out to be hugely rewarding and exciting for all the participants.

Particular key functions of the team meetings are not only to actually work on the pending projects. On a secondary level the team enjoys a continuing dialogue between mathematical creativity and intuition and artistic creativity and intuition: A dialogue of “How to take steps towards the unknown?” and of “How are these steps taken in different fields?”

On a tertiary level the team also experiences elements of current art-philosophy, as discussed, for instance, in Epistemologies of Aesthetics by Dieter Mersch [14], who deconstructs and displaces the terminology that typically accompanies the question of the relationship between art and scientific truth. Identifying artistic practices as modes of thought that do not make use of language in a way that can easily be translated into scientific discourse, Mersch advocates for an aesthetic mode of thought beyond the linguistic turn, a way of thinking that cannot be substituted by any other disciplinary system.

Besides all these aspects of the communication within the team members, the discourse of the team meetings is also fluently shifting within the following triangle of dialogues:



Moments of inner-mathematical discussions morph into phases of inner-artistic discourses and both feed into the work on the tasks of the Mathematics and Arts project, which currently comprise the following three topics:

- Cluster algebras, triangulations and frieze patterns, see Section 3 below
- Models of collective behaviour
- Models with "wavy" entropy decay.

So far, the biggest part of the project focussed on algebra and combinatorics of triangulations, which is the subject of the following Section 3.

Concerning models of collective behaviour, we are in particular interested in models, where the swarming behaviour can be controlled by the energy of the swarm. Since serving artistic inspiration is the only purpose for these models, the mathematicians are here given the liberty of studying objects just for the sake of them "looking good".

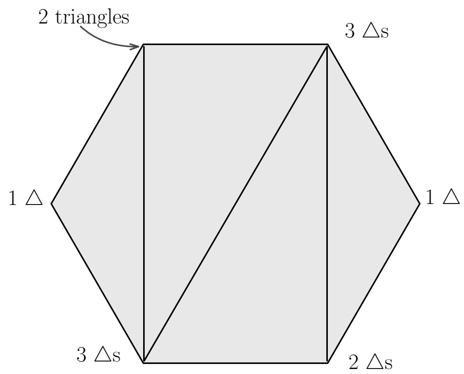
Models with "wave" entropy are models, where the interaction between (a complex) microscopic and (a controllable) macroscopic scale leads to a dynamical system, which allows evaluation in terms of suitable entropy functionals.

3 Cluster algebras, triangulations and frieze patterns

Around 2000, Fomin and Zelevinsky [8] invented cluster algebras – certain rings of variables, generated by possibly infinitely many overlapping sets of generators. The authors were motivated by phenomena observed in the study of dual canonical bases of enveloping algebras and of total positivity of matrices/of algebraic groups. Zelevinsky's beautiful description in [15] is recommended as a first introduction to the topic. Fomin and Zelevinsky showed in [9] how cluster algebras arise from triangulations of polygons. This approach has been developed further since then, notably by Fomin, Shapiro and Thurston in [7]. Under this correspondence, cluster algebras give rise to frieze patterns of positive numbers, if one specialises the cluster variables in a cluster to 1. Frieze patterns have first been studied by Coxeter [5] and then Conway and Coxeter [3] in the 70's. A frieze pattern consists of sequences of rows of integers, forming a lattice as in Figure 2, starting with a row of zero's, then a row of one's, and then several rows of positive integers, such that in any diamond formed by four entries, the product of the horizontal neighbours is one plus the product of the vertical neighbours:

$$\left.\begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array}\right\} \implies d := \frac{bc-1}{a} \in \mathbb{N}.$$

The frieze is finite if it ends after finitely many rows with a row of one's followed by a row of zero's. It is infinite otherwise. Conway and Coxeter showed in [3] that every finite frieze is invariant under a glide reflection and that it arises through the triangulation of a (regular) polygon: the first non-trivial row is given as the so-called matching numbers for the triangulation, i.e., the number of triangles incident with the vertices of the polygon. The glide symmetry implies in particular that the frieze is periodic, with period a divisor of the size of the polygon. The frieze of Figure 2 arises from a snake triangulation of a hexagon as in the figure here. The matching numbers are 2,1,3,2,1,3 (going counterclockwise around the polygon).



Infinite friezes arise from triangulations of annuli, where the arcs have endpoints on both boundaries of the figure [2]: the first non-trivial row is given by the matching numbers for a triangulation of an annulus (with marked points on both boundaries).

Through discussions in the group, we have started various projects related with triangulations of surfaces. On one hand, the observation of the fast growing entries in infinite friezes led to a collaboration on the growth of such patterns [1].

On the other hand, some conceptual similarities between calculating the matching numbers forming an n-periodic frieze and models of n interacting individuals led

0		0		0		0		0		0		0
	1		1		1		1		1		1	
2		**1**		**3**		**2**		**1**		**3**		**2**
	1		2		5		1		2		5	
2		1		3		2		1		3		2
	1		1		1		1		1		1	
0		0		0		0		0		0		0

Figure 2: A six- (and three-) periodic finite frieze pattern arising from a hexagon.

the team also to look for possible entropy functionals, which might be useful to describe the growth behaviour of infinite frieze patterns. And indeed, by reinterpreting formulas of matching numbers of *n*-periodic friezes in terms of probabilities of observing first- and higher order relationships between vertices, we were able to identify a class of *Shannon-type entropy functionals*, which seem to characterise the growth behaviour of infinite friezes. In fact, these entropy functionals evaluate line-wise *n*-periodic frieze patterns, by taking (roughly speaking) the expected value of the information content associated to the relationship probabilities and average these over *n* neighbouring entries of the frieze.

Interestingly, while the entries of frieze patterns are not necessarily line-wise monotone, the decay of these Shannon-type entropy functional appears to be strictly line-wise monotone and approximately exponentially decaying towards an entropy minimum as depicted in Figure 3.

The detailed properties of these entropy functionals and their possible use to characterise frieze patterns are currently under investigation. The discovery of the

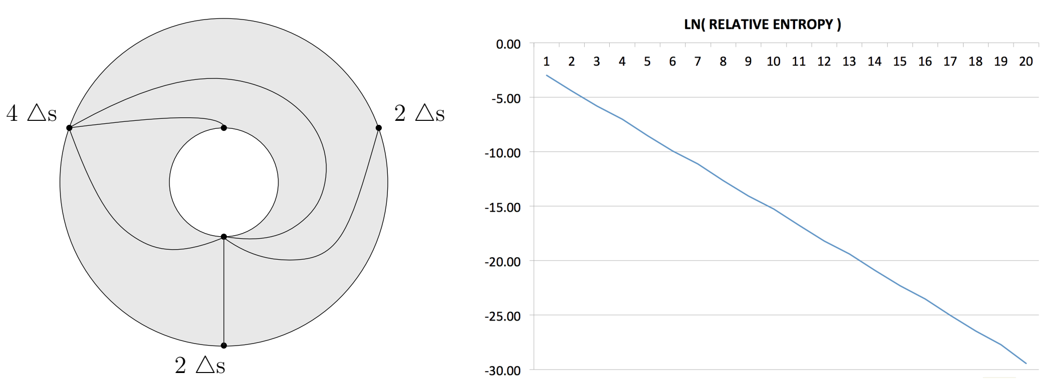


Figure 3: Decay of relative a Shannon-type entropy functional for an infinite frieze pattern derived from a topological triangulation of an annulus.

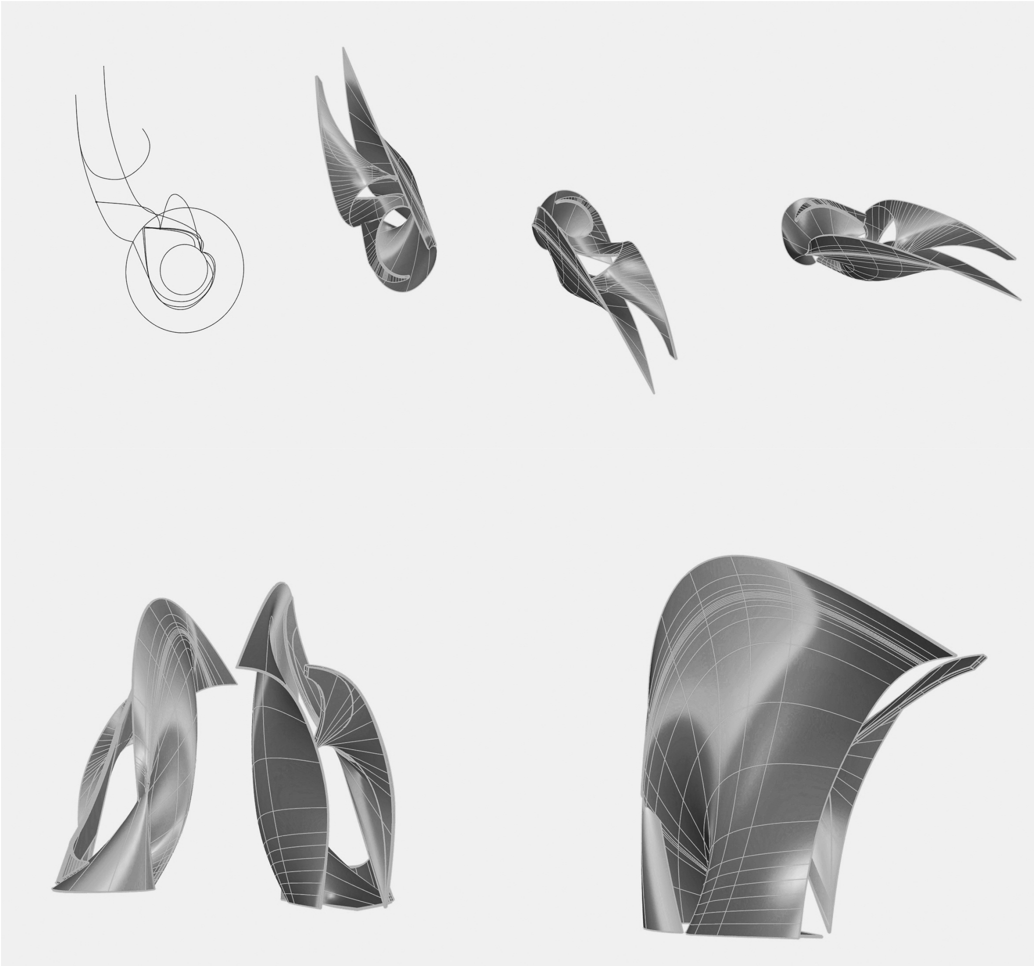


Figure 4: Exploring proportions and relations of triangulations.

existence of such functionals was only made possible by cross-referencing mathematical intuitions from algebra and partial differential equations and constitutes an inner-mathematical example of the creation of new ideas within our project.

4 Attractive Privacies of Breathing Borders

Alongside the inner-mathematical discussions on frieze patterns, Tamara Friebel became also artistically interested in properties of triangulations and has developed her personal interpretation into two musical compositions. As a preliminary study, she derived 3d objects, which served as test cases to illustrate possible proportions and relations, see e.g. Figure 4.

The composition “Attractive Privacies of Breathing Borders” by Tamara Friebel

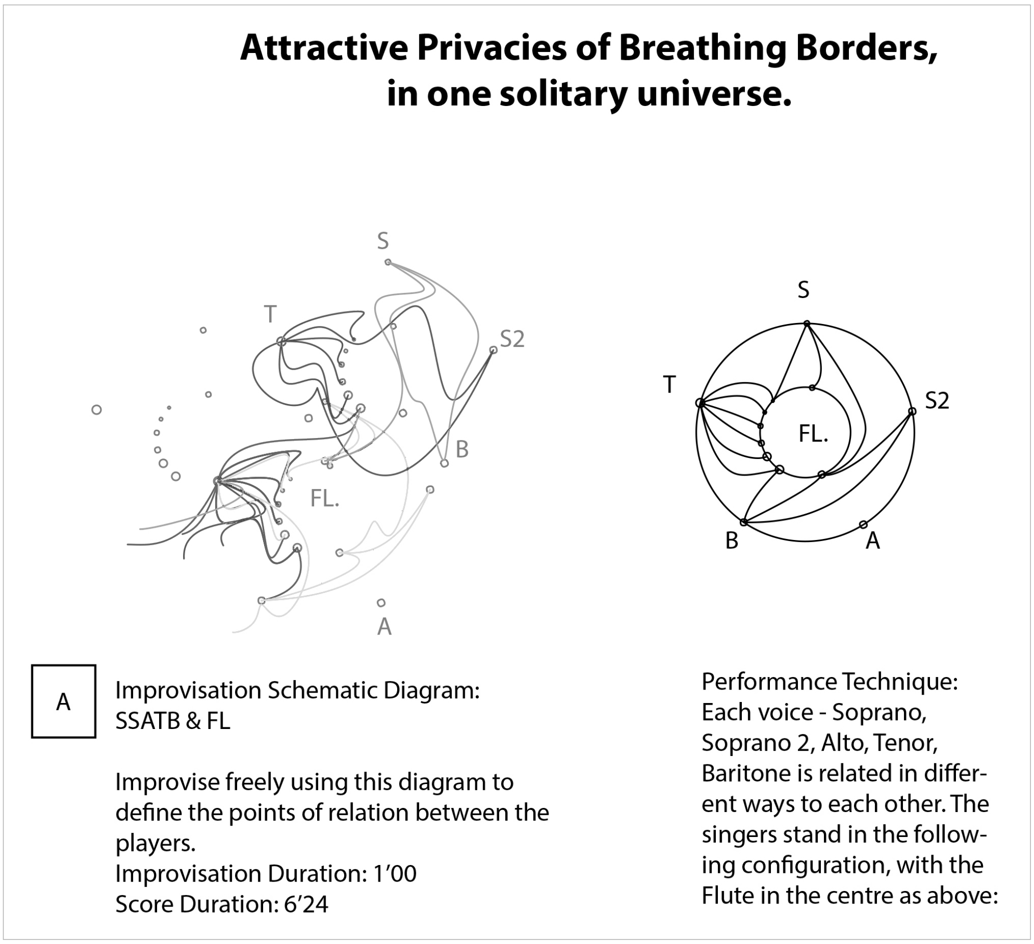


Figure 5: “Attractive Privacies of Breathing Borders” (SSATB + flute, 2015), excerpt of the concept sheet.

for flute and five vocal soloists (SSATB) was presented at the SALT festival at the University of Victoria, Canada. The piece is drawn from the mathematical research of Section 3. Mathematical elements like triangulations became hereby artistic elements like connectivity and transparency of borders: Constraints, which Friebel uses in the composition process.

Figures 5 and 6 provide excerpts of the cover sheet and the score of *Attractive Privacies of Breathing Borders*. A triangulation of an annulus is not only taken for the staging of the vocalists around the central flute, but served also as a starting point to derive communication and separation patterns between the six parts.

In the first performance of the piece, the musicians were asked as a first part to improvise several constellations represented by graphical scores. The musicians were so given the opportunity to explore individual associations. Then, as a sec-

Figure 6: "Attractive Privacies of Breathing Borders" (SSATB + flute, 2015), beginning of the score.


12
cresc and catch upper octave and surrounding harmonics
Fl.
S1
S2
A
T
B
Oo
m
16
gradually move from tone - to rich harmonics
--to upper whistle tones
gradually move from tone - to rich harmonics
--to upper whistle tones

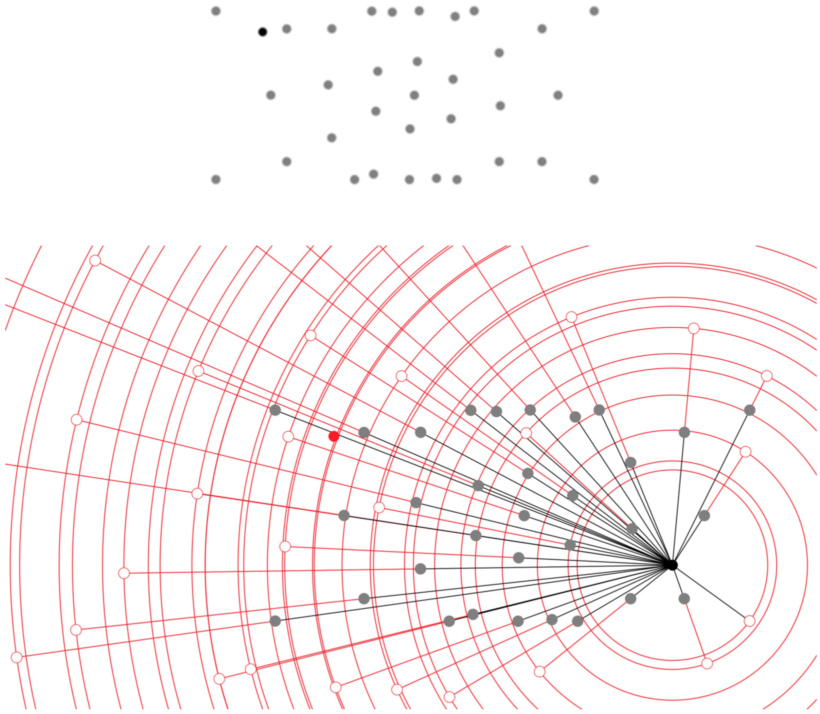


Figure 7: Speaker arrangement and sound wave alignment of *Zeitraum*.

ond part, the performance of the composer's score followed.

A second composition „Zwielichter, herumtanzend" (Twilights, dancing around us) for baroque flute and harpsichord was first performed at the Essl Museum Klosterneuberg in December 2015 (Ensemble Klingekunst, see [13] for a preview on YouTube). 'Zwielichter, herumtanzend" continued to explore schematic diagrams based on topological triangulations, which were then intuitively read and transcribed into individual "strings" of music. In a second step these strings were then further layered and juxtaposed.

5 Outlook

The current work within the project continues to explore the three main topics, (i) triangulations and friezes (ii) models of collective behaviour and (iii) models with "wavy" entropies.

The project team is moreover currently contemplating formats of a workshop event to be organised in fall, which shall be able to reflect and stimulate the many different aspects of the project: mathematical and artistic research, artistic practice and performances, educational aspects, ...

A further project concerns variants of the sound installation *Zeitraum* by Gerhard Eckel [6]. Zeitraum is a sound environment exposing the interrelation of time and space in acoustic communication, composed of many identical sound sources dispersed irregularly in a large space. When listened to from a particular location, the so called sweet spot, the pattern is perceived as an isochronous pulse. When distancing oneself from the sweet spot, then physical as well as psycho-acoustic effects compel the listener's brain to "compose" a personal sound experience.

And finally, we shall always wholeheartedly recommend any mathematician traveling Scotland, to never do so without an artist. Because it is really exciting to explore the full range of possibilities of one-sided sheep.

6 Acknowledgments

The authors are supported by the Research Project "Mathematics and Arts: Towards a balance between artistic intuition and mathematical complexity" funded 2014-2017 by the University of Graz.

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Website of the project: *http://thecollaborativemind.com/*

Internat. Math. Nachrichten
Nr. 231 (2016), 15–33

Persistent Homology – State of the art and challenges

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1 Motivation for multi-scale topology

A recurring task in mathematics, statistics, and computer science is understanding the connectivity information, or equivalently, the topological properties of a given object. For concreteness, we assume the object in question to be a geometric shape, possibly embedded in a high-dimensional space, although that assumption is not necessary for most of the theory. Algebraic topology offers a toolset for quantifying and comparing topological features of such shapes.

The strongest notion of topological equivalence, the existence of an *homeomorphism* between topological spaces, is out of reach in general in computational contexts.[1] An attractive compromise is offered by the theory of homology over a base field $\mathbb{F}$. In informal terms, the p-th homology group $H_p(\mathcal{S})$ of a shape $\mathcal{S}$ (with $p \geq 0$) is a $\mathbb{F}$-vector space whose rank counts the number of "p-dimensional holes" in $\mathcal{S}$. Concretely, for objects embedded in $\mathbb{R}^3$, $\operatorname{rank} H_{0,1,2}(\mathcal{S})$ count the number of connected components, tunnels, and voids, respectively, induced by the shape $\mathcal{S}$.

Homology over fields reveals less topological information then the $\mathbb{Z}$-homology, but this partial information is sufficient for many purposes. The main advantage of restricting to fields is the existence of efficient algorithms. More precisely, if the input is given as a combinatorial cell complex, the homology groups in all dimensions can be computed in cubic time with respect to the number of cells.

[1]The question whether two shapes are homeomorphic is undecidable for shapes of dimension 4 and higher [54].

ISSN 0020-7926

Multi-scale and noise. We discuss three basic exemplary scenarios in which topological data reveal potentially valuable information. For each scenario, other tools can be employed as well; the goal is rather to underline the general applicability of topology as a tool for data analysis.

- Combustion is a highly complex dynamic process relevant for engineering applications. Consider the goal of analyzing the temperature distribution of a combustion for a fixed moment in time. One approach could be to fix a temperature threshold and decompose the domain into "hot" and "cold" areas. The connectivity of these areas allows an identification of hot or cold pockets which might guide the analyst to areas of importance in the process.

- The task of shape retrieval is to find for a query point cloud (for instance obtained by a 3D-scanner) the closest representation in some database of shapes. A topology-based similarity measure provides a high-level summary which can be used to quickly rule out shapes with very different topology.

- Clustering is one of the most fundamental problems in data analysis. As an example, imagine an internet company collecting data about users in terms of various real-valued parameters. The users form a high-dimensional point cloud, and grouping them into clusters of similar users facilitates decision making (e.g., personalized product offers) and predictions of the user's behavior in the future. Understanding the topology of that "user space" can be helpful to design a reasonable notion of similarity measures.

The combustion example above contains a scale parameter, identifying what parts are considered hot and cold. A parameter is also intrinsic in the other applications: at first sight, the input is merely a discrete point cloud without interesting topological features. It is required to build a model of the underlying space from which the point cloud was drawn (i.e., the shape that has been scanned). The most frequently employed technique is to replace the points by balls of a fixed radius, and to take the union of these balls as an approximation of the underlying space (cf. Figure 1). In this case, the ball radius constitutes the scale parameter. This raises the question of which radius to choose: a small radius might give a too fine-grained picture while a large radius might blur relevant information contained in the data.

In many applications, there is no natural choice of what is the best scale to look at. In such cases, one might want to consider various scales and to select the best choice afterwards. However, this *multi-scale* approach is affected by the presence of *noise* in the data. For instance, an inaccurate scanning of a shape might lead to a large number of "bubbles" in the approximation, increasing the number of voids in the shape and occluding the real topological features. Such noise can be present at all scales, complicating the task of separating signal and noise in the data.

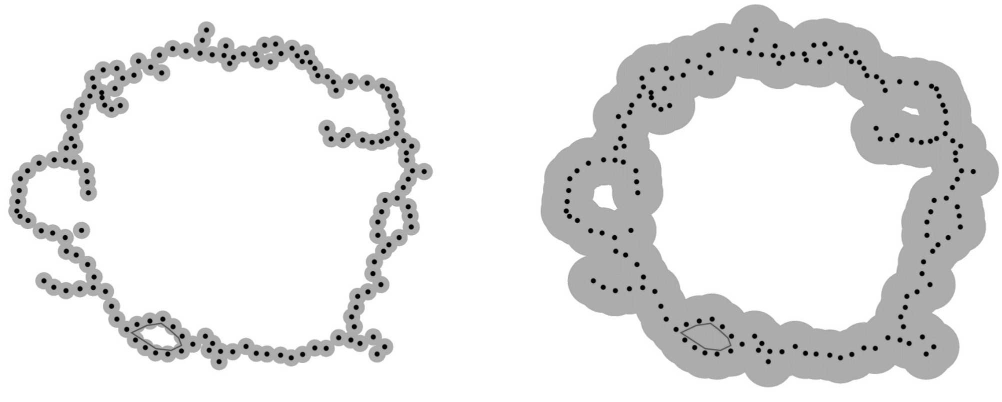


Figure 1: Representation of a point cloud on two different scales as a union of balls. On the smaller scale, we count 5 holes in the shape, or equivalently, $\beta_1 = 5$. On the larger scale, $\beta_1 = 3$. However, the *persistent* Betti number of the inclusion map is 1, because 4 of the 5 small-scale holes disappear after the inclusion. This is illustrated for the bottom hole by the blue cycle generating the corresponding homology class, which becomes trivial in the larger union. Only the larger hole "survives" the inclusion from small to large scale, making it the only persistent feature that spans over this range of scales.

Persistent homology. The main idea of persistence is to connect the homological information gathered across different scales. In this way, we can identify which topological features are present over a large range of scales as opposed to those which are only spuriously present.

To describe the idea mathematically, consider two spaces $X \subseteq Y$, corresponding to representations of data on different scales (think about two sublevel sets of a function, or two unions of balls with different radius). The inclusion map $X \hookrightarrow Y$ induces, for any $p \geq 0$, a linear map between the vector spaces

$$\phi : H_p(X) \to H_p(Y),$$

as a consequence of the functorial properties of homology [56]. We define the *persistent Betti number* with respect to (X,Y) as

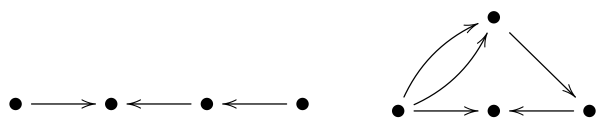
$$\operatorname{rank}(\operatorname{Im}\phi),$$

which counts the number of homological features in Y which have already been present in X (see Figure 1 for an example). Having a multi-scale representation of a given data set, we obtain a persistent Betti number for each pair of scales. They constitute a topological multi-scale summary of the data, which provides more information than only the ranks of the individual homology groups. A catchy one-liner for this idea is that "the homology of a sequence is worth more than a sequence of homologies" [41].

2 Quivers and Barcodes

Under some mild assumptions, there are effective ways to visualize the persistent homology of a sequence. They are called *persistence diagrams* or *barcodes*. We describe the latter using notions from representation theory. The content of this section is a shortened version of the recent exposition by Oudot [58].

Quivers and representations. A *quiver* is a directed multigraph with *nodes* and *arrows*. A quiver is called *finite* if both the number of nodes and arrows is finite. Here are two examples of quivers



(1)

A finite quiver is called A_n*-type*, if after removing all its arrowheads, it takes the form:

$$\bullet \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \cdots \!-\!\!-\!\!-\! \bullet \!-\!\!-\!\!-\! \bullet \tag{2}$$

For a fixed quiver Q with node set V and arrow set A and a base field $\mathbb{F}$, a *representation* $V = ((V_i)_{i\in V}, (f_{ij})_{ij\in A})$ is an assignment of a $\mathbb{F}$-vector space V_i to each node i of Q and a linear map $f_{ij} : V_i \to V_j$ to each arrow from i to j. There are no further conditions on the resulting diagram of vector spaces and linear maps, in particular, the maps do not have to commute. A representation is called *finite-dimensional*, if $\dim V_i < \infty$ for all nodes i. The simplest example of a representation is the trivial one, assigning the trivial vector space to every node.

Our motivating example originates from a sequence

$$S_1 \hookrightarrow S_2 \hookrightarrow \ldots \hookrightarrow S_{n-1} \hookrightarrow S_n$$

of growing shapes, for example representing a given data set for scales $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. Applying $\mathbb{F}$-homology for fixed dimension p yields a sequence of vector spaces and linear maps

$$H_p(S_1) \xrightarrow{h_1} H_p(S_2) \xrightarrow{h_2} \ldots \xrightarrow{h_{n-1}} H_p(S_{n-1}) \xrightarrow{h_n} H_p(S_n) \tag{3}$$

which is a representation of an A_n-type quiver with all arrows directed to the right. While we focus on finite quivers in this article, the theory can be extended to the infinite case as explained in [58].

Having two representations V and W of the same quiver, we can form another representation $V \oplus W$ naturally by taking the direct sums of vector spaces and linear maps over every node and arrow. Vice versa, we call a representation V *indecomposable*, if $V = W_1 \oplus W_2$ implies that W_1 or W_2 is the trivial representation.

Decompositions. Let us consider the simplest quiver $\bullet$, consisting of one node and no arrow. A finite-dimensional representation is simply a finite-dimensional vector space, and thus isomorphic to $\mathbb{F}^k = \mathbb{F} \oplus \ldots \oplus \mathbb{F}$ for some k. Thus, every representation decomposes into a unique direct sum of indecomposable elements up to isomorphism, and the only indecomposable representation is $\mathbb{F}$. For more general quivers, it turns out that the former statement remains valid, while the classification of indecomposable elements is more involved.

Before we can state the result, we have to define isomorphisms of representations in general. A *morphism* ϕ between two representations $V = (V_i, f_{ij})$ and $W = (W_i, g_{ij})$ of the same quiver Q is a collection of linear maps $\phi_i : V_i \to W_i$ such that for any arrow from i to j in Q, the diagram

$$\begin{array}{ccc} V_i & \xrightarrow{f_{ij}} & V_j \\ \downarrow{\phi_i} & & \downarrow{\phi_j} \\ W_i & \xrightarrow{g_{ij}} & W_j \end{array} \tag{4}$$

commutes. A morphism is called *isomorphism*, if each ϕ_i is an isomorphism of vector spaces. The following theorem, attributed to Krull, Remak, and Schmidt, settles the existence and uniqueness of a decomposition of finite representations.

Theorem 1. *Let V be a non-trivial, finite-dimensional representation of a finite quiver. Then, $V = V_1 \oplus \ldots \oplus V_k$, where each V_i is non-trivial and indecomposable. This decomposition is unique up to permutations and isomorphism.*

What are the indecomposable representations of a quiver? It turns out that for A_n-type quivers, the situation is well-behaved. This result is due to Gabriel [39].

Theorem 2. *Let V be an indecomposable, finite-dimensional representation of an A_n-quiver. Then, V is isomorphic to the representation $I_{b,d}$, with $1 \leq b \leq d \leq n$, which is*

$$\underbrace{0 \overset{0}{\text{---}} \cdots \overset{0}{\text{---}} 0}_{b-1} \overset{0}{\text{---}} \underbrace{\mathbb{F} \overset{id}{\text{---}} \cdots \overset{id}{\text{---}} \mathbb{F}}_{d-b+1} \overset{0}{\text{---}} \underbrace{0 \overset{0}{\text{---}} \cdots \overset{0}{\text{---}} 0}_{n-d}$$

In particular, every representation satisfying the requirements of the theorem can be characterized as a finite collection of intervals. We call this collection of intervals the *barcode* of the representation.

Persistent barcodes. What do these results imply for the homology sequence in (3)? A simple observation is that the barcode reveals the Betti number of $H_p(S_i)$ for all i, just by counting the number of intervals that span over i. But equally, the persistent Betti numbers are also encoded in the barcode: for $i < j$, let $\beta_{ij} = \operatorname{rank}\operatorname{Im} f$, where $f : H_p(S_i) \to H_p(S_j)$ is induced by the inclusion map $S_i \hookrightarrow S_j$. By functoriality, $f = h_{j-1} \circ \ldots \circ h_i$, and consequently, β_{ij} equals the number of intervals in the barcode that span over the whole range $[i, j]$. Vice versa, the persistent Betti numbers also uniquely determine the barcode: the number of indecomposables of the form $I_{b,d}$ is given by

$$\beta_{b,d} - \beta_{b-1,d} - \beta_{b,d+1} + \beta_{b-1,d+1} \tag{5}$$

by the inclusion-exclusion principle.

The intervals in the barcode can also be interpreted in intuitive geometric terms: it is instructive to imagine the sequence $S_1 \hookrightarrow \ldots \hookrightarrow S_n$ as a sequence of growing balls with a fixed set of centers. Setting $p = 2$, the barcode captures the formation of voids in this sequence of balls. An interval $[b, d]$ means that a new void comes into existence when the balls have reached the scale α_b. This void persists until scale α_d where it is completely filled up, and disappears. Similar considerations are true for tunnels ($p = 1$), and connected components ($p = 0$). Figure 2 illustrates this idea for an example in the plane.

While barcodes can be defined without the use of the rather heavy machinery of quivers (for instance, using (5)), this abstract point of view has several advantages: First of all, it underlines that the concept of persistence is rather independent of homology and applies to sequences of vector spaces in general (with $\mathbb{F}$-homology being only one instance of it). More importantly, we obtain a non-trivial generalization for free. Consider the following example of a *zigzag sequence* of spaces

$$S_1 \hookrightarrow S_2 \hookrightarrow S_3 \hookleftarrow S_4 \hookrightarrow S_5 \hookleftarrow S_6.$$

We can interpret this sequence again in the context of data analysis, allowing cases where the approximation is allowed to expand or shrink when the scale parameter increases. Functoriality of homology now yields a sequence of homology groups and linear maps

$$H_p(S_1) \to H_p(S_2) \to H_p(S_3) \leftarrow H_p(S_4) \to H_p(S_5) \leftarrow H_p(S_6)$$

in the same way as before. Because the arrows point in different directions, the concept of persistent Betti numbers does not carry over to this context. However, the homology groups still form a representation of a A_n-type quiver. Therefore, Theorem 2 applies also to this case and ensures the existence of a barcode!

Finally, the representation-theoretic point of view sheds some light on the theory of *multidimensional persistence*, where one considers more than one scale parameter to analyze the data set. The complete version of Gabriel's theorem [39] shows that finding a compact description of persistent homology in more than one dimension becomes a delicate issue; we will discuss this in some more detail in Section 4.

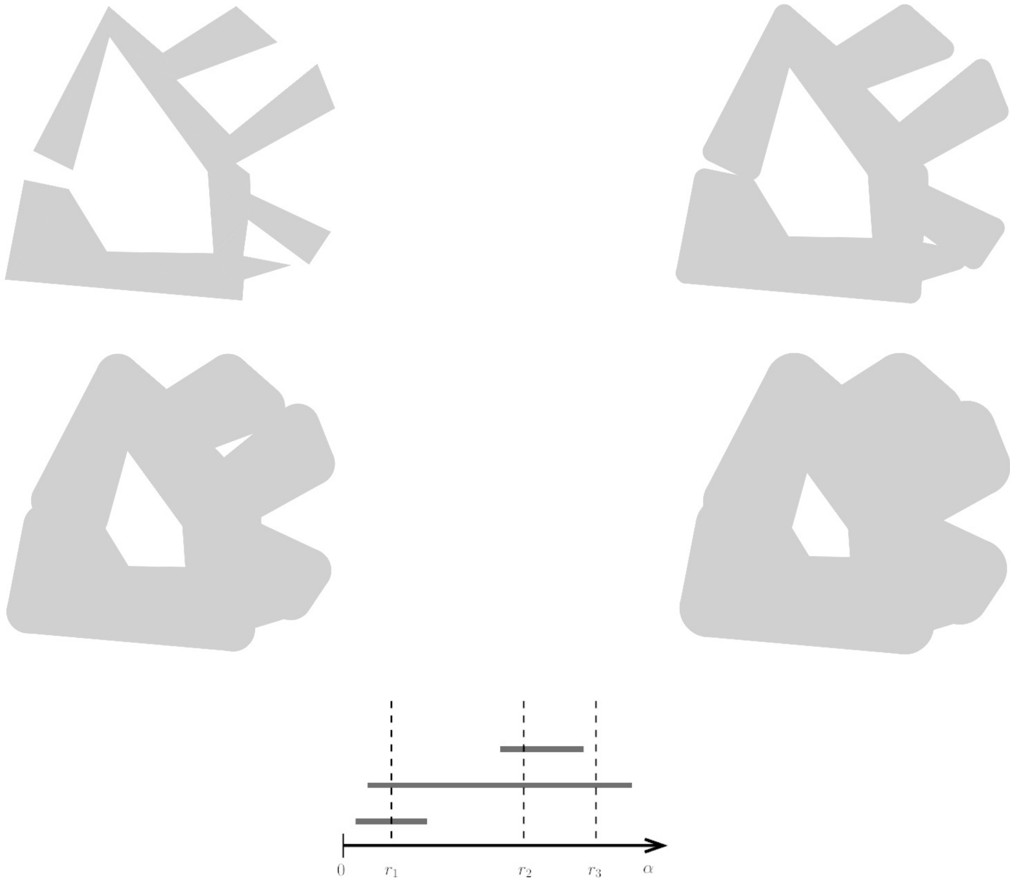


Figure 2: These 4 images show snapshots of a nested sequence of shapes $S_1 \hookrightarrow S_2 \hookrightarrow \ldots \hookrightarrow S_{n-1} \hookrightarrow S_n$. Observe the formation and vanishing of holes in this process. The barcode summarizes this process. Each bar (i.e., indecomposable) corresponds to a hole in the process and spans over the range of scales for which the hole is present in the data. The vertical alignment of the bars is not important. This illustration already appeared in [45].

3 About the history of persistence

Although persistent homology only exists for about 15 years in the literature, the substantial amount of work makes a comprehensive survey a difficult task. Moreover, any such attempt is doomed to be deprecated within short time due to the rapid evolvement of the research field. We therefore do not even aim for completeness, but rather focus on a few highlights in the theory, applications and algorithmic aspects of persistent homology. The interested reader can find more details in one of the numerous surveys on the topic [40, 10, 32, 67, 35, 66, 27]. There are also various textbooks available covering persistent homology [33, 31, 68, 41, 58].

Theory The term "persistent homology" was coined by Edelsbrunner, Letscher, and Zomorodian [34], who introduced persistent Betti numbers, persistence diagrams (a different, but equivalent representation of barcodes) and an efficient algorithm for filtrations of alpha shapes in the case $\mathbb{F} = \mathbb{Z}_2$. Zomorodian and Carlsson [69] extended this algorithm to arbitrary fields; moreover, they provided an algebraic description of persistence as a graded $\mathbb{F}[t]$-module, and argued that all persistent Betti numbers are determined uniquely by the module decomposition. The connection of persistence to quiver theory, as described earlier, was introduced by Carlsson and de Silva [11] to develop the concept of *zigzag persistence*.

A cornerstone for the importance of persistence is its *stability*: it means that a small perturbation of data leads only to a small change in its barcode summary; to make the statement precise, a distance measure on barcodes has to be defined, which we omit in this article. Cohen-Steiner, Edelsbrunner, and Harer [24] provided the first such stability result for the so-called *bottleneck distance*, and this result was extended by Cohen-Steiner et al. [25] to a wider family of distance measures. Once again abstracting from the geometric context, stability has been rephrased in algebraic terms by the concept of *interleavings* by Chazal et al. [16].

The survey by Carlsson [10] discusses many of these aspects and also popularized the idea of using the theory of persistent homology as a general technique for data analysis tasks. This has led to the shapening of *topological data analysis (tda)* as a new research discipline in which persistent homology is a key concept. We point out that tda is a wider area, covering aspects that are not discussed in this article, including size theory [38], Morse-Smale complexes [44], sheaf theory [26], and Reeb graphs [5]. We remark that an extension of Reeb graphs, the Mapper algorithm [63] forms the basis of the startup company Ayasdi[2], underlining the relevance of topological tools in industrial applications of data analysis.

Applications There is a large bandwidth of application scenarios on which persistent homology has been proved to be useful. A comprehensive list goes beyond the scope of this article, but we mention applications in coverage problems in sensor networks [28], measuring the dimension of fractal shapes [53], robust length measuring of tube-like shapes [36], the analysis of growth of rice plant roots [4], the effect of mixture of genome material in evolution [15], the effects of drug influence on brain networks [59], and the visualization of cyclical behavior of memory assignments in the execution of machine programs [23]. The recent book by Oudot [58, p.8] contains a longer (and mostly disjoint) list. We point out that the last three mentioned applications deal with data of non-geometric nature, but the data still has "shape" for which topology reveals meaningful information.

Among the numerous applications, we illustrate two major templates of how topological information is used by describing two applications in slightly more detail:

[2] *http://www.ayasdi.com*

Chazal et al. [20] consider the problem of clustering point clouds. Among the many approaches for this problem, *mode-seeking* methods [50] construct a density function f based on the point cloud, create one cluster center per local minimum, and cluster the point set using the *basins of attraction* for each minimum (with respect to the gradient flow). A problem with this method is the instability of the clustering under small perturbation of f, and the authors use persistent homology to tackle this problem: using the persistent barcode defined by the function f, they classify the clusters into important ones and noisy ones, based on the range of scales in which a cluster is active. Then, they employ a robust variant of mode-seeking clustering where the basins of noisy clusters are charged to important ones; see [20] for more details. This is an example of a *denoising*: the topological internals of a particular data set are analyzed, allowing a simplified and more robust outcome for the given task (this was also the original motivation of introducing persistent homology from [34]).

The second template of applications uses topological information as a proxy in order to compare and classify data sets. The majority of contemporary applications falls in this category. An instructive example is given by Adcock, Rubin, and Carlsson [1], who study the task of classifying images of liver lesions into predefined categories, for the purpose of computer-assisted diagnosis. For that, they compute a barcode on an image, and compute the pairwise distances of that barcode to the barcode of a set of reference images. This defines a high-dimensional feature vector, where each coordinate is based on a topological distance. Having represented an image as a high-dimensional point, the authors use standard techniques from machine learning, such as support vector machines, for the classification task, and report on satisfying results. While this result approaches the classification task solely on topological descriptors, topology can also be used to complement other (e.g., geometric) descriptors [42, 64].

Algorithms A major reason for the success of persistent homology as a discipline is the existence of fast algorithms to compute the topological summary. For computations, the multi-scale representation of the data is usually written as an inclusion of combinatorial cell complexes, and is represented by the *ordered boundary matrix* of that cell complex. Persistence is computed by a simple reduction procedure that resembles Gaussian elimination. While its theoretical worst-case complexity is cubic in the size of the matrix, the algorithm shows a significantly better behavior in practice, thanks to the initial sparseness of the boundary matrix.

Because of the demand for practically efficient implementations, there is a substantial body of literature describing speed-ups of the original matrix reduction. One line of research attempts to identify shortcuts in the reduction process exploiting the special structure of boundary matrices, and achieves remarkable speed-ups with rather simple heuristics [2, 21]. These techniques have also lead to the first practical distributed algorithm to compute persistent homology [3]. Also successful has been the approach of computing persistent *cohomology* instead, relying on a duality result for persistent homology and cohomology by de Silva et al. [29]. Boissonnat et al. [6] provided several optimizations of the original algorithm under the name of *annotations* [30]. Yet another way of improving is the combination of Discrete Morse Theory and persistence [43, 55]: the idea is to reduce the size of the initial simplicial complex through collapses guided by a Morse matching, and to invoke the matrix reduction algorithm solely on a matrix representation of the collapsed complex, which is often of significantly smaller size. All the aforementioned techniques have been implemented in publicly available software packages – we refer to [57] for a recent comparative survey.

The standard problem of comparing two barcodes can be reduced to a maximum-cardinality matching problem in complete bipartite graphs [33, §VIII.4]. It has been observed recently that the special (geometric) structure of barcodes can be used to significantly speed-up these computations in practice [47].

4 Current developments

Persistent homology has shown to be a useful tool to analyze data sets under a topological lens. Nevertheless, many questions remain unanswered both in terms of generalization and scalability. We end this article by highlighting three areas of active research which have the potential to significantly extend the range of applications of the theory.

Multidimensional persistence A limitation of standard persistent homology is the restriction to a single scale parameter. In many applications, one would like to filter the data along two or more axes: for instance, in the combustion example from before, we would probably prefer to consider a time-varying sequence of functions measuring temperature, and to track topological changes for progress in time as well as for changes in the threshold.

The simplest formalization of this process is a diagram of spaces and maps

$$
\begin{array}{ccccccc}
S_{m1} & \hookrightarrow & S_{m2} & \hookrightarrow & \ldots & \hookrightarrow & S_{mn} \\
\uparrow & & \uparrow & & & & \uparrow \\
\vdots & & \vdots & & & & \vdots \\
\uparrow & & \uparrow & & & & \uparrow \\
S_{21} & \hookrightarrow & S_{22} & \hookrightarrow & \ldots & \hookrightarrow & S_{2n} \\
\uparrow & & \uparrow & & & & \uparrow \\
S_{11} & \hookrightarrow & S_{12} & \longrightarrow & \ldots & \longrightarrow & S_{1n}
\end{array}
\qquad (6)
$$

where all little squares commute (the time-varying example above would better be modeled by a zigzag diagram, but we try to keep the exposition simple). Applying homology yields a representation of the quiver whose shape is the integer grid.

How much of the theory for one dimension carries over? Theorem 1 from Section 2 applies to the quiver, stating that the representation decomposes into finitely many indecomposables. However, Theorem 2 only holds for A_n-type quivers (and slight generalizations of it). The structure of indecomposables is considerably more complicated in general: there is an infinite number of isomorphisms classes, already for the case of a square-shaped quiver, which prevents a direct generalization of barcodes to higher dimensions. These difficulties with the multidimensional case have been observed first by Carlsson and Zomorodian [12] (without using quiver theory).

Despite these negative results, multidimensional persistence has received growing attention in the last years. While a complete topological invariant like the barcode in one dimension is out of reach, the primary question is which incomplete invariants can be useful for the data analysis applications. The first proposal was the *rank invariant* [12] which generalizes the persistent Betti numbers: in two dimensions, it is defined as

$$\text{rank}\left(H_p(S_{ij}) \to H_p(S_{k\ell})\right)$$

for any $i \leq j, k \leq \ell$. Cerri et al. [13] have considered one-dimensional sections of the multi-dimensional filtration. In the setting of (6), any monotone path from S_{11} to S_{mn} defines a one-dimensional barcode, and the collection of all these barcodes is equivalent to the rank invariant. Very recently, Lesnick and Wright [52] developed a software to visualize this collection of barcodes, along with improved algorithms to compute the rank invariant.

Another research front is the efficient comparison of multidimensional representations. Lesnick [51] extended the *interleaving distance* to the multidimensional case. Chacholski et al. [14] proposed an formal algebraic definition of noise and define the distance between two representation as the minimal noise in which they differ. While both approaches are mathematically sound, no efficient algorithms to compute or at least approximate these distances are known, and no hardness results have been settled.

Because of the demand for analyzing data in multi-dimensional scale spaces, we expect further research to define, compute, visualize, and compare meaningful invariants for the case of multidimensional persistence.

Statistical tda A recent line of research is the combination of persistent homology and statistical methods. A central question in this context is the definition of an average of a collection of diagrams. Difficulties arise from the fact that the space of persistent barcodes has a complicated structure; while so-called *Fréchet means* of barcodes can be defined in this space, they are not unique and difficult to compute [65]. An alternative idea is to embed the space of barcodes into a larger and better behaved space, in which means are well-defined and simple to compute.

We have already discussed an example of such a strategy for the diagnosis of liver lesions [1] in Section 3. Recall that the barcode of an image was converted into a point in $\mathbb{R}^d$, constituting a transition into standard Euclidean space for which a large toolset of statistical methods applies. Another concept is that of *persistent landscapes* by Bubenik [8]. A persistence barcode is converted into a sequence of functions $\ell_i : \mathbb{R} \to \mathbb{R}$. Having two or more landscapes, averaging is easily achieved through a pointwise average of the i-th level functions. However, the average landscape in general cannot be translated back to a persistence barcode. Landscapes satisfy basic statistical properties such as a law of large numbers and a central limit theorem, and standard statistical methods like bootstrapping [18, 19] and subsampling [17] have been brought into the field of topological data analysis. Yet another approach by Reininghaus et al. [60] defines a kernel for persistence barcodes which induces a Hilbert space structure on barcodes and permits topological classifiers in machine learning applications, such as Support Vector Machines and Principle Component Analysis. Two recent software libraries provide methods to apply statistical methods on persistence diagrams [9, 37].

We foresee further applications of statistical methods in the analysis of realistic data sets. Besides a comparison of existing techniques to embed the barcode space, plenty of algorithmic challenges need to be resolved: how can we efficiently compute and represent such an embedding? What are meaningful statistical tests, and how can they be performed efficiently in the context of persistence?

Efficient creation of cell complexes The first step in the computational pipeline of persistent homology is the generation of a sequence of shapes, representing the

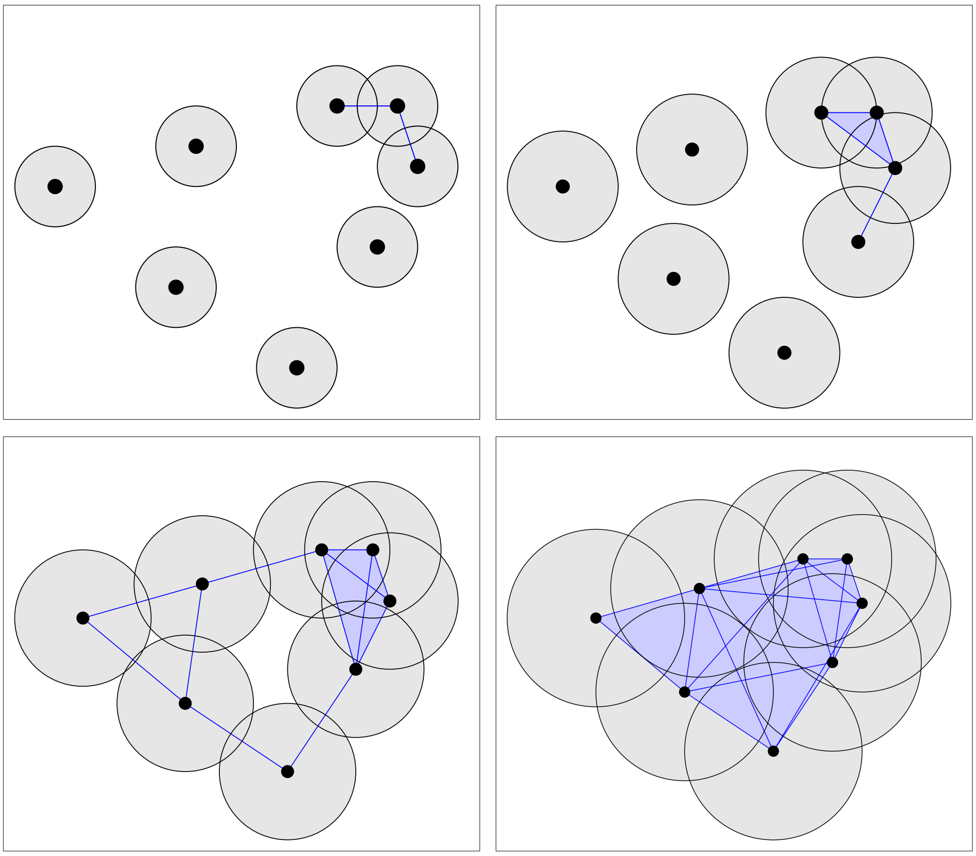


Figure 3: Illustration of the Čech filtration as the intersection complex of a union balls at various scales.

input data on different scales. We remind the reader of the popular example of point clouds, and their approximation by a union of balls, whose radius increases throughout the sequence. For computational purposes, it is common to dualize the construction, and to consider the *nerve* of the balls, which is a simplicial complex that captures the intersection patterns of the balls, called the Čech complex (Figure 3). The major drawback is the sheer size of this complex: for n points in $\mathbb{R}^d$, it grows to a size of

$$O(n^{d+1})$$

simplices – this is too much for realistic applications already when d is small.

For low dimensions, especially $d = 2$ and $d = 3$, the complex size can be reduced by the use of *alpha complexes* [33], forming a subset of the Delaunay triangulation of the point set. But this improvement does only slightly improve the asymptotic bound for high dimensions (to $O(n^{\lceil d/2 \rceil})$) and raises computational questions since computing Delaunay triangulations in high dimensions is a non-trivial task.

A promising direction is to use geometric approximation techniques to approximate cell complexes: instead of computing a homotopically equivalent representation of desired shapes, the goal is to find approximate complexes which are significantly smaller in size, but with a provable guarantee of closeness of the exact and approximate persistent barcode. Sheehy [61] gave the first construction for the related *Vietoris-Rips* complexes with a size of

$$O(n \cdot 2^{d^2})$$

(the precise bound is more fine-grained, but we restrict to the worst-case estimate for brevity) for an arbitrary fixed constant approximation quality ε. Similar results for Rips and Čech complexes with the same asymptotics have been derived subsequently [7, 30, 49]. Because of the decoupling of n and d in the bound, these techniques have the potential to broaden the range of data sets for which persistence can be applied. The practical evaluation of these techniques is one of the major challenges of algorithmic topology within the next years.

There is also a line of research dealing with very high-dimensional input (i.e., if d is in the same order as n). In this case, the approaches mentioned above do not improve the naive construction. Instead, dimension reduction techniques have been considered. The celebrated Johnson-Lindenstrauss lemma [46] states that a point cloud in high-dimensionsal Euclidean space can be embedded into

$$O(\log n)$$

dimensions with arbitrary small distortion. As shown by Sheehy [62] and by Kerber and Raghvendra [48], this property extends in the following way: the Čech complex of a point set in high dimensions yields a persistent barcode that is close to the barcode of the same point set projected to $O(\log n)$ dimensions.

Very recently, Choudhary, Kerber, and Raghvendra [22] developed a new approximation technique that yields an approximation complex with only

$$O(n \cdot 2^{d \log d})$$

simplices, at the price of a weaker approximation guarantee. Combined with dimension reduction techniques, their results yield an approximation complex whose size is

$$n^{O(1)},$$

independent of the dimensionality d of the point set.

"Big data" is one of the buzzwords of our time – how can we design algorithms that are able to cope with the increasing volume of acquired data? Approximation techniques appear to be the most promising paradigm to process the immense amounts of data in a reasonable time. The aforementioned efforts can be interpreted as an attempt of transferring these technique into the context of tda. The question of how far this transfer will go has to be carried out by research in the upcoming years.

Acknowledgments I thank Aruni Choudhary and Hannah Schreiber for their helpful suggestions.

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Stellenausschreibung UniversitätsassistentIn mit Doktorat für 6 Jahre Technische Universtät Graz, Institut für Diskrete Mathematik

1 Stelle eines/einer Universitätsassistenten/in mit Doktorat für 6 Jahre, 40 Stunden/Woche, voraussichtlich ab 1. Oktober 2016.

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Gewünschte Qualifikationen: Wissenschaftliche Arbeiten auf den Gebieten der Zufallsprozesse (random walks), Graphentheorie, geometrischen Gruppentheorie, nach Möglichkeit übergreifend zwischen diesen Themenkreisen. Bereitschaft zur Mitarbeit in diesbezüglichen Forschungsprojekten. Lehrtätigkeit im Umfang von mindestens 4 Wochenstunden pro Semester, insbesondere in der Ingenieurmathematik (deutschsprachig).

Für weitere Informationen siehe *http://www.math.tugraz.at/∼woess/position*

Bewerbung, Lebenslauf und weitere Unterlagen sind unter Nennung der Kennzahl 5050/16/009 bis 31. Mai 2016 an die TU Graz, Dekan der Fakultät für Mathematik, Physik und Geodäsie, Prof. Wolfgang Ernst, Petersgasse 16, A-8010 Graz zu richten, bevorzugt per E-Mail an: ⟨*bewerbungen.mpug@tugraz.at*⟩.

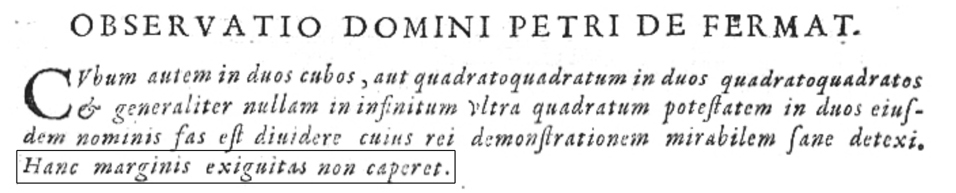
Internat. Math. Nachrichten
Nr. 231 (2016), 35–38

Der Rand war zu schmal

Johannes Wallner
TU Graz

Am 24. Mai 2016 wird Kronprinz Haakon von Norwegen den diesjährigen Abelpreis an Andrew Wiles verleihen, der im Jahr 1993 durch die Veröffentlichung eines Beweises für die Fermatsche Vermutung: *„Die Gleichung $a^n + b^n = c^n$ besitzt keine Lösung mit ganzzahligen $a, b, c > 0$ und ganzzahligem Exponenten $n > 2$“* weltweite Bekanntheit erlangte. Die spannende Geschichte der Fertigstellung des Beweises durch Andrew Wiles gemeinsam mit Richard Taylor [6, 5] ist wohlbekannt, genauso wie die Historie der Vermutung selbst: Pierre de Fermat brachte ca. im Jahre 1640 in seinem Exemplar der lateinischen Übersetzung der antiken *Arithmetik* des Diophant von Alexandria 48 Randbemerkungen an. Seinem Sohn ist es zu verdanken, dass diese in eine spätere Ausgabe derselben Übersetzung aufgenommen wurden und so auf uns gekommen sind. Eine davon enthält Fermats Vermutung samt Behauptung, einen Beweis dafür zu haben (siehe Abbildung 1).

Mit dem heutigen Wissen über die Hintergründe können wir mit an Sicherheit grenzender Wahrscheinlichkeit ausschließen, dass Fermat tatsächlich einen korrekten Beweis besessen hat. Er hat jedoch den Spezialfall $n = 4$ erfolgreich behan-


OBSERVATIO DOMINI PETRI DE FERMAT.

CVbum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum poteſtatem in duos eiuſdem nominis fas eſt diuidere cuius rei demonſtrationem mirabilem ſane detexi. Hanc marginis exiguitas non caperet.

„Dagegen ist es unmöglich, einen Kubus in zwei Kuben, oder ein Biquadrat in zwei Biquadrate, und allgemein irgendeine Potenz außer dem Quadrat in zwei Potenzen von demselben Exponenten zu zerlegen. Ich habe einen wahrhaft wunderbaren Beweis dafür entdeckt, den dieser enge Rand jedoch nicht fassen würde.“

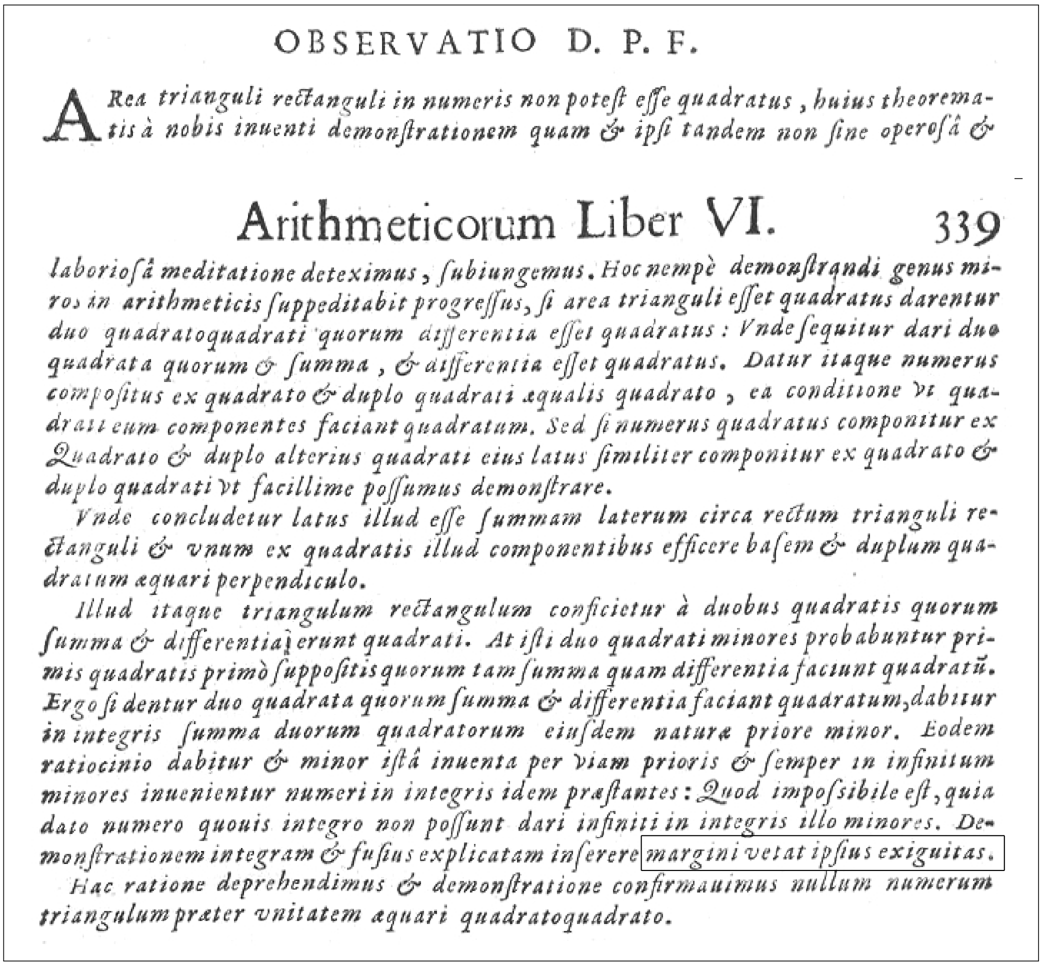
Abbildung 1: Kommentar von P. Fermat, publiziert in der von seinem Sohn C. S. Fermat besorgten Ausgabe der *Arithmetik* des Diophant [1, p. 61], direkt nach Frage 8 im 2. Buch der Arithmetik. Leider war der Rand zu schmal, um Fermats Beweis der Unmöglichkeit von $a^n + b^n = c^n$ für $n > 2$ auch nur anzudeuten.

ISSN 0020-7926

delt. Es handelt sich dabei um den einzigen Beweis, den uns Fermat hinterlassen hat. Er zeigte nicht nur die Unlösbarkeit von $a^4 + b^4 = c^4$, sondern sogar von

$$c^4 - b^4 = x^2. \qquad (1)$$

(Mit $x = a^2$ ergibt sich $a^4 + b^4 = c^4$, siehe Abb. 2 und 3.) Seine Methode des *descente infinie* ist äquivalent zum Induktionsprinzip. Es dürfte weniger bekannt sein, dass sich Fermat im Beweis von (1) ebenfalls über die Enge des Randes beklagt. Er findet jedoch Platz, um – mit vollem Recht – zu behaupten, dass die von ihm präsentierte Methode (i.w. die Methode der vollständigen Induktion) wunderbare Fortschritte in der Arithmetik ermöglichen wird. Ihre Publikation in [1]


OBSERVATIO D. P. F.

Area trianguli rectanguli in numeris non potest esse quadratus, huius theorematis à nobis inuenti demonstrationem quam & ipsi tandem non sine operosâ &

Arithmeticorum Liber VI. 339

laboriosâ meditatione deteximus, subiungemus. Hoc nempè demonstrandi genus miros in arithmeticis suppeditabit progressus, si area trianguli esset quadratus darentur duo quadratoquadrati quorum differentia esset quadratus: Vnde sequitur dari duo quadrata quorum & summa, & differentia esset quadratus. Datur itaque numerus compositus ex quadrato & duplo quadrati æqualis quadrato, ea conditione vt quadrati eum componentes faciant quadratum. Sed si numerus quadratus componitur ex Quadrato & duplo alterius quadrati eius latus similiter componitur ex quadrato & duplo quadrati vt facillime possumus demonstrare.

Vnde concludetur latus illud esse summam laterum circa rectum trianguli rectanguli & vnum ex quadratis illud componentibus efficere basem & duplum quadratum æquari perpendiculo.

Illud itaque triangulum rectangulum conficietur à duobus quadratis quorum summa & differentia erunt quadrati. At isti duo quadrati minores probabuntur primis quadratis primò suppositis quorum tam summa quam differentia faciunt quadratū. Ergo si dentur duo quadrata quorum summa & differentia faciant quadratum, dabitur in integris summa duorum quadratorum eiusdem naturæ priore minor. Eodem ratiocinio dabitur & minor istâ inuenta per viam prioris & semper in infinitum minores inuenientur numeri in integris idem præstantes: Quod impossibile est, quia dato numero quouis integro non possunt dari infiniti in integris illo minores. Demonstrationem integram & fusius explicatam inserere margini vetat ipsius exiguitas.

Hac ratione deprehendimus & demonstratione confirmauimus nullum numerum triangulum præter vnitatem æquari quadratoquadrato.

Abbildung 2: Kommentar von P. Fermat zum 20. Problem, das der Übersetzer C. G. Bachet zum 6. Buch der *Arithmetik* des Diophant hinzugefügt hat [1, p. 338f]. Der Rand war zu schmal, um Fermats Beweis der Unmöglichkeit von $c^4 - b^4 = x^2$ und damit auch von $a^4 + b^4 = c^4$ vollständig widerzugeben. Das Prinzip des *descente infinie* ist jedoch ausführlich dargelegt.

ungefähr 30 Jahre nach der ersten privaten Niederschrift durch den Autor erfolgte fast zeitgleich mit der systematischeren Behandlung der Induktion durch Pascal und Bernoulli. H. Edwards gibt in [3, §1.6] eine detaillierte Kritik dieses Beweises. Fermat hat ihn nie veröffentlicht, aber mehrmals brieflich versucht, Kollegen zu Beweisen seiner Aussage herauszufordern.

In insgesamt vier seiner 48 Randnotizen erwähnt Fermat die Beengtheit (*exigui-*

„Die Fläche eines rechtwinkeligen Dreiecks, dessen Seiten rationale Zahlen sind, kann keine Quadratzahl sein. Den Beweis dieses von mir gefundenen Satzes habe ich selbst erst durch mühevolles und eifriges Nachdenken entdeckt. Ich lasse den Beweis hier folgen, da diese Art der Beweisführung wunderbare Fortschritte in der Arithmetik ermöglichen wird.

Wenn die Fläche eines rechtwinkeligen Dreiecks eine Quadratzahl wäre, so gäbe es zwei Biquadrate, welche eine Quadratzahl zur Differenz hätten. Es würde folglich zwei Quadratzahlen geben, deren Summe sowohl wie Differenz ein Quadrat wäre. Daher würden wir eine Quadratzahl haben, welche gleich der Summe eines Quadrats und des Doppelten eines Quadrats wäre, während zugleich die beiden Quadrate, aus denen sie gebildet ist, selbst eine Quadratzahl zur Summe hätten. Wenn aber eine Quadratzahl in ein Quadrat und das Doppelte eines zweiten Quadrats zerfällt werden kann, [so ist] auch ihre Seite gleich der Summe eines Quadrats und des Doppelten eines Quadrats [wie man sehr leicht zeigen kann].

Daraus schließen wir, dass diese Seite die Summe der Katheten eines rechtwinkeligen Dreiecks ist, dass nämlich das einfache Quadrat, welches sie enthält, die Basis, das doppelte Quadrat das Lot ist.

Dieses rechtwinkelige Dreieck wird somit von zwei Quadraten gebildet, deren Summe und Differenz Quadrate sein werden. Aber diese beiden Quadrate sind, wie sich zeigen lässt, kleiner als die ersten anfangs angenommenene Quadrate, deren Summe und Differenz Quadrate sind. Wenn es also zwei Quadrate gibt, deren Summe und Differenz Quadrate sind, so gibt es auch zwei andere ganze Quadratzahlen von derselben Beschaffenheit wie jene, welche aber eine kleinere Summe haben. Durch dieselben Schlüsse findet man, dass es eine noch kleinere Summe als die vermittels der ersteren gefundene gibt, und so werden ins Unendliche fort immer kleinere ganze Quadratzahlen gefunden werden, welche dasselbe leisten. Das ist aber unmöglich, weil es nicht unendlich viele ganze Zahlen geben kann, welche kleiner sind als eine beliebig gegebene ganze Zahl. Den Beweis ganz und ausführlicher hier mitzuteilen, dazu reicht der Rand nicht aus.

Durch diese Überlegung habe ich auch gefunden und bewiesen, dass keine Dreieckzahl außer 1 ein Biquadrat sein kann.“

Abbildung 3: Übersetzung nach [2] der Randnotiz von P. Fermat aus Abbildung 2.

tas) des Randes. Vielleicht ist es kein Zufall, dass zwei mit großer Bedeutung dabei sind: Die eine (Abbildung 1) verspricht einen Beweis, der sicherlich falsch war. Die Suche nach diesem Beweis war für sehr viele Personen Inspiration und Frustration und erst gegen Ende des 20. Jahrhunderts durch eine außerordentliche und innovative Anstrengung erfolgreich, als die Zahlentheorie und algebraische Geometrie entsprechend weit fortgeschritten waren.

Die andere Randnotiz (Abb. 2, 3) enthält den einzigen von Fermat selbst überlieferten Beweis, der zwar lückenhaft ist, aber auf einem innovativen und vom Autor zeitlebens geheim gehaltenen Beweisprinzip beruht. Man kann vermuten, dass Fermat dessen Bedeutung erahnt hat.

Passenderweise spielt in der Geschichte der Fermatschen Vermutung die *Arithmetik* des Diophant eine große Rolle. Dieses einzige nicht verloren gegangene algebraische Werk des antiken Griechenland ist in 6 Büchern im griechischen Original und vier weiteren in arabischer Übersetzung überliefert, die erst in den 1970-er Jahren wieder aufgefunden wurden [4].

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Buchbesprechungen

L. Bukovský: The Structure of the Real Line. (Monografie Matematyczne, Vol. 71). Birkhäuser, Basel, 2011, xiv+536 S. ISBN 978-3-0348-0005-1 H/b € 99,95.

Das Buch beginnt im ersten Kapitel mit einer kurzen Einführung in die Zermelo-Fraenkel-Mengenlehre. In den darauf folgenden Kapiteln (2–8) werden für die Maßtheorie und Topologie wichtige Eigenschaften von Teilmengen der reellen Linie bewiesen. Dabei legt der Autor großen Wert darauf, wenn möglich das Auswahlaxiom zu vermeiden. So sind Sätze, die mithilfe des Auswahlaxioms (oder einer schwächeren Version des Auswahlaxioms) bewiesen wurden, entsprechend gekennzeichnet.

Die letzten zwei Kapitel des Buchs beschäftigen sich dann mit fortgeschrittenen Themen der Mengenlehre. So wird im Kapitel 9 die Abhängigkeit der weiteren Axiome (wie die Kontinuumsjypothese, Martins Axiom, usw.) untersucht, und man findet dazu im Buch große Abhängigkeitsdiagramme. Das letzte Kapitel untersucht dann die Unentscheidbarkeit gewisser Eigenschaften.

Die Ergebnisse der letzten zwei Kapitel beruhen auf vielen Resultaten, die mittels der Forcing-Methode bewiesen wurden. Da aber eine Einführung in Forcing den Rahmen des Buchs sprengen würde, diskutiert der Autor alle für das Buch relevanten Ergebnisse in einem Anhang, die mit Hilfe von Forcing in den letzten 50 Jahren bewiesen wurden.

Alles in allem liefert das Buch einen sehr gelungenen Überblick und auch einen sehr tiefen Einblick in die Interaktion zwischen Analysis und Mengenlehre. Das Buch ist gut lesbar und bis auf das Kapitel zu Forcing ist der Autor auch sehr darauf bedacht, das Buch in sich selbst abgeschlossen zu halten.

V. Ziegler (Salzburg)

J. R. Faulkner: The Role of Nonassociative Algebra in Projective Geometry. Graduate Studies in Mathematics, Vol. 159). American Mathematical Society, Providence, Rhode Island, 2014, xiv+229 S. ISBN 978-1-4704-1849-6 H/b $ 67 € 54,–.

Der trefflich gewählte Titel dieses Buchs umschreibt präzise die Intention des Autors. Sein Ziel ist eine weitestgehend in sich geschlossene und zugleich in die Tiefe gehende Darstellung der Querverbindungen zwischen nicht-assoziativen Algebren und projektiven Ebenen unter besonderer Berücksichtigung der Ebenen über Oktaven. So wie im Buch soll das Attribut *nicht-assoziativ* auch hier als *nicht notwendig assoziativ* verstanden werden. Das Werk ist in 14 Kapitel gegliedert, von denen jedes mit einem Ausblick beginnt und durch zahlreiche (ergänzende) Aufgaben abgeschlossen wird. Mit Ausnahme des letzten Kapitels sind alle Beweise komplett angegeben, womit sich das Buch auch als Einführung für Studierende in ein aktuelles Forschungsgebiet eignet.

In den ersten drei Kapiteln finden wir Altbekanntes: Axiomatisch definierte affine und projektive Ebenen, Zentralkollineationen projektiver Ebenen und die Koordinatisierung projektiver Ebenen mithilfe von Ternärringen; Letztere werden als *ternery systems* bezeichnet. Das anschließende Kapitel 4 liefert die grundlegenden Struktursätze für alternative (Divisions-)Ringe. In Kapitel 5 werden die Konfigurationen von Desargues und Pappos sowie der Satz vom Vierseitschnitt untersucht. Im Mittelpunkt steht dabei deren Rolle als geometrische Seitenstücke zu gewissen Identitäten in nicht-assoziativen Ringen. Kapitel 7 ist der Theorie projektiver Räume gewidmet, wobei vorbereitend in Kapitel 6 eine allgemeine Hüllen- und Dimensionstheorie vorgestellt wird. Mittels Zentralkollineationen wird die unverzichtbare Brücke zur linearen Algebra geschlagen: Jeder projektive Raum mit projektiver Dimension ≥ 3 ist zu einem Vektorraum-Modell (über einem assoziativen Divisionsring) isomorph. In ähnlicher Weise wird in Kapitel 8 der Fundamentalsatz der projektiven Geometrie hergeleitet, also die algebraische Beschreibung der Kollineationen von projektiven Räumen über Vektorräumen. Die anschließenden Kapitel 9, 10 und 11 sind quadratischen Formen, homogenen Abbildungen, Normen und Hermiteschen Matrizen gewidmet. Mit deren Hilfe gelingt dann in Kapitel 12 eine elegante Beschreibung der projektiven Ebenen über Oktaven. Verallgemeinerte projektive Ebenen mit Distanzrelation (*remoteness relation*) sind das Thema von Kapitel 13. Beispiele werden aus freien Moduln vom Rang 3 über assoziativen Ringen mit Eins gewonnen. Bei der Untersuchung der Transvektionen dieser Ebenen ergeben sich schöne Querverbindungen zu den Gruppen vom Steinberg-Typ. Das abschließende Kapitel 14 bringt einen Ausblick in allgemeinere Geometrien, wie Tits-Gebäude und verallgemeinerte n-Ecke.

H. Havlicek (Wien)

R. Haller, F. Barth: Berühmte Aufgaben der Stochastik von den Anfängen bis heute. De Gruyter Oldenbourg Verlag, München, 2014, xvi+448 S. ISBN 978-3-486-72832-3 H/b € 79,95.

Das vorliegende Werk ist eine umfangreiche Sammlung stochastischer und kombinatorischer Problemstellungen von der Steinzeit bis zur Gegenwart. Sprunggelenksknochen von Tieren (sog. Astralogoi) wurden von der Steinzeit bis in die Spätantike als Orakel verwendet, indem sie wie Würfel geworfen wurden. Glücksspiele mit Würfeln waren im Mittelalter verbreitet. In der Renaissance beschäftigten sich die Menschen schon mit anspruchsvolleren kombinatorischen und stochastischen Problemen. Im 18. Jahrhundert veränderte das Entstehen der Analysis die Welt der Stochastik, es wurden Grenzwerte betrachtet oder Integrale zu Hilfe genommen. Bei den dargestellten Problemen der jüngeren Vergangenheit stehen nicht ungelöste mathematische Fragen im Vordergrund, sondern die Unterhaltung. Manche der Lösungen sind verblüffend, sie widersprechen der Intuition und sind somit besonders lehrreich.

Die Beispiele dieser Sammlung sind nicht nur gut dokumentiert, sondern oft auch mit historischen Illustrationen versehen. Interessant ist das Werk als Quelle für alle, die Grundkurse in Stochastik oder Kombinatorik halten, ob in der Schule oder an den Universitäten. Für Gymnasiallehrkräfte ist das Buch eine gute Quelle für sogenannte Typ 2-Beispiele, bei denen längere Textangaben üblich sind und Querverbindungen hergestellt werden müssen.

B. Krön (Wien)

P. M. Higgins: Das kleine Buch der Zahlen. Vom Abzählen bis zur Kryptographie. A. d. Englischen übersetzt von T. Filk (Springer Spektrum) Berlin 2013 xii+354 S. ISBN 978-3-8274-3015-1 S/b € 20,51.

Eines gleich vorweg: „Das kleine Buch der Zahlen“ zu lesen, macht großen Spaß. Das Buch von Higgins richtet sich in erster Linie an Mathematik-Interessierte und erzählt in leichter und unterhaltsamer Form eine Geschichte der Zahlen vom Zahlenbegriff und den Fragestellungen der alten Ägypter und Griechen bis zu den modernen Anwendungen der Mathematik in der Public Key-Kryptographie. Jedes Kapitel ist voll von interessanten Fakten, überraschenden Beispielen, gelösten und offenen Problemen, persönlichen Schicksalen und amüsanten Anekdoten, und es ist spannend von der ersten bis zur letzten Seite.

Der Autor beschäftigt sich zunächst ausführlich mit den natürlichen Zahlen, er erzählt von Primzahlen, Carmichael- und Ackermann-Zahlen, vom Pascalschen Dreieck, magischen Quadraten und anderen Zahlentricks. Es folgen negative, gebrochene, irrationale und schließlich komplexe Zahlen, verbunden mit einer ausführlichen und spannenden Geschichte der Entwicklung der Algebra von Cardano bis Abel, Galois und Gauß. Verbindungen zu Problemen des Alltags findet man besonders in zwei Anwendungsschwerpunkten, einem Kapitel über Zufall und Wahrscheinlichkeit und einem Abschnitt über Kryptographie.

Erst im letzten Kapitel für „Kenner und Feinschmecker“ beschreibt der Autor in 57 Anmerkungen mathematische Details und Hintergründe zum zuvor Gesagtem. So geht er etwa ein auf Rekursionen für Binomialkoeffizienten und Stirling-Zahlen, die Abzählbarkeit der Menge der rationalen und algebraischen Zahlen oder auf die Irrationalität und die Kettenbruchdarstellungen von e. Und für alle, bei denen der Funke übergesprungen ist, gibt es zum Abschluss ausführlich kommentierte Literaturempfehlungen.

G. Karigl (Wien)

H. Iwaniec: Lectures on the Riemann Zeta Function. (University Lecture Series 62) American Mathematical Society, Providence, Rhode Island, 2014, vii+119 S., ISBN 978-1-4704-1851-9, P/b $ 40 € 37,–.

The book focuses on a proof of an approximation to the Riemann Hypothesis of N. Levinson that at least 34 percent of the zeros of the Riemann zeta function have real part $1/2$.

The first part of the book covers classical material about the zeros of the Riemann zeta function with applications to the distribution of primes. The second part describes completely Levinson's method.

The book is based on lecture notes given in Rutgers in 2012. The book is recommended to all readers interested in Riemann's Hypothesis and accessible after a first course in complex analysis with a little knowledge of analytic number theory.

A. Winterhof (Linz)

P. Roquette: Contributions to the History of Number Theory in the 20th Century. (EMS Heritage of European Mathematics). EMS, Zürich, 2013, xiv+189 S. ISBN 978-3-03719-113-2 H/b € 78,–.

Der (neu edidierte) Nachdruck von elf Aufsätzen ergibt eine interessante Darstellung der Entwicklung und Entstehung der modernen Algebra und Zahlentheorie. Die Akteure treten uns anhand von Dokumenten und Briefen lebendig entgegen. Dass auch bedeutende Mathematiker vor Fehlern in ihren Beweisen nicht geschützt sind, kann viel Sympathie erwecken. Bemerkenswert sind die Ergebnisse von Otto Grün, der 1935 an Helmut Hasse schreibt: „Ich habe meine Kenntnisse nur aus Büchern geschöpft ...“. Von den vielen Namen seien nur einige weitere genannt: Cahit Arf, Richard Brauer, Klaus Hoechsmann, Heinrich-Wolfgang Leopoldt, Emmy Noether, Abraham Robinson, Ernst Steinitz und Hermann Weyl. Dennoch muss ein kleines Caveat ausgesprochen werden: Die Lektüre erfordert an vielen Stellen gute mathematische Kenntnisse. Wer aber Lehrveranstaltungen zu diesen Themen hält oder diese mit Interesse besucht, wird einigen Gewinn davon haben. Das Buch kann aber auch Einsicht in die Zeitgeschichte geben, berührt es doch Fragen wie die Diskriminierung von Frauen in der Wissenschaft oder die politische Lage in Deutschland. Bemerkenswert ist, dass Otto Grün noch 1948 in Berlin promovieren konnte, obwohl er niemals Student war, was heute die universitäre Bürokratie wohl „erfolgreich“ verhindern würde.

F. Schweiger (Salzburg)

D. A. Salamon: Funktionentheorie. (Grundstudium Mathematik). Birkhäuser, Basel, 2012, viii+218 S. ISBN 978-3-0348-0168-3 P/b € 24,95.

Dieses neue Lehrbuch entstand aus Vorlesungsaufzeichnungen des Autors zu einer einsemestrigen Vorlesung an der ETH Zürich. Das als Einführung gedachte Buch basiert inhaltlich auf L. Ahlfors Buch „Complex Analysis“ und setzt Grundkenntnisse über Analysis und Lineare Algebra voraus.

Die Auswahl des Stoffs orientiert sich naturgemäß an der zur Verfügung stehenden Präsentationszeit, ist aber breit genug, um dem Leser einen fundierten Einstieg in dieses interessante und sehr wichtige Gebiet der Mathematik bereitzustellen. Die fünf Kapitel (Komplexe Zahlen, Holomorphe Funktionen, Integralformel von Cauchy, Residuenkalkül, Riemannscher Abbildungssatz) werden nahezu gänzlich in der Vorlesung des Autors behandelt. Drei umfassende Anhänge über harmonische Funktionen, zusammenhängende Räume und kompakte metrische Räume stellen Inhalte bereit, die in der Vorlesung an vielen Stellen Verwendung finden, dort aber nicht im Detail vorgetragen werden.

Das Buch ist sehr klar verfasst und bereitet Freude beim Lesen bzw. Durcharbeiten. Gut gelungen sind die mehrfarbigen Abbildungen. Zahlreiche Übungsbeispiele ermuntern den Lernenden zur eigenständigen Beschäftigung mit dem Stoff.

E. Werner (München)

S. Serfaty: Coulomb Gases and Ginzburg-Landau Vortices. (Zurich Lectures in Advanced Mathematics) EMS, Zürich, 2015, viii+157 S. ISBN 978-3-03719-152-1 P/b € 34,00.

This slim book of approximately 150 pages is based on lecture notes for a graduate course, taught by the author at the ETH Zrich in Spring 2013. It is mainly based on the authors own research. Its two main topics are the mathematical description of point particles interacting via Coulomb potential forces, a typical example being the classical Coulomb gas, and the study of vortices in the famous Ginzburg-Landau model of superconductivity. Coulomb gases are important models in statistical mechanics with close relations to random matrix theory. At low temperatures, these models are expected (and partly known) to yield concentration effects on so-called Fekete sets, which exhibit a lattice structure. It is this feature which ties Coulomb gases to the study of vortices in superconducting materials, since also in the latter triangular lattice structures emerge, the so-called Abrikosov lattices, once the material is subject to an external magnetic field of critical strength.

The book describes these two systems and explores both their similarities and differences. The presentation is self-contained and introduces the mathematical tools (e.g., Gamma-convergence, large deviations, renormalization techniques, free-boundary value problems, etc.) necessary to analyze these systems. The results and proofs are presented in a neat and thorough way, with lots of additional remarks linking to closely related topics in the literature.

The author is a renowned expert on these fields, and the book consequently presents the current state-of-the-art results available within the mathematical literature. The book should be helpful to researchers of all levels who are either already working in these fields, or who want to make themselves familiar with these research topics.

C. Sparber (Chicago)

T. Tao: Hilbert's Fifth Problem and Related Topics. (Graduate Studies in Mathematics, Vol. 153). American Mathematical Society, Providence, Rhode Island, 2014, xiii+338 S. ISBN 978-1-4704-1564-8 H/b \$ 69,–, € 62,–.

Das 5. Problem von Hilbert ist eine Aussage der Form: „Eine lokal-euklidische topologische Gruppe ist isomorph zu einer Liegruppe." (Man beachte, dass bei einer Liegruppe Gruppenverknüpfung und Inversion beliebig oft differenzierbar sind, bei einer topologischen Gruppe hingegen nur als stetig vorausgesetzt werden.) Der vorliegende Band behandelt die Methoden, die zur Lösung dieses Problems führen sowie die beiden folgenden Themen:

(a) Eine *approximative Gruppe* A ist eine Teilmenge einer Gruppe, sodass „viele" Produkte von Elementen aus A wieder in A sind. Dieses Konzept spielt in der additiven Kombinatorik eine Rolle. T. Tao diskutiert die Klassifikation der approximativen Gruppen in nichtabelschen Gruppen.

(b) Eine endlich erzeugte Gruppe hat *polynomielles Wachstum*, wenn die Anzahl der Elemente der Länge n (bezüglich einer symmetrischen Erzeugermenge) höchstens $O(n^d)$ ist, für ein festes d. Das Buch behandelt den Satz von Gromov, der besagt, dass eine endlich erzeugte, polynomiell wachsende Gruppe G eine nilpotente Untergruppe G' hat, sodass $|G/G'|$ endlich ist.

Dieser Band ist aus einer Graduiertenvorlesung entstanden und enthält viele Übungsaufgaben. Der Autor stellt zahlreiche Querverbindungen zu anderen Gebieten her und seine Erklärungen trennen sehr gut das gedanklich Wesentliche von technischen Details. Zusätzlich hat der Autor die Vorlesungen parallel auf seinem Blog verfügbar gemacht. Auf diese Weise hat er bereits zahlreiche Rückmeldungen und Korrekturen von Lesern erhalten und in den Text einbauen können.

Dem Autor gelingt es, einmal pro Jahr ein Buch wie dieses vorzulegen, und er allein verfasst damit seit 6 Jahren ca. 10 Prozent der bekannten Buchreihe *Graduate Studies in Mathematics* der AMS, nämlich die Nummern 117 (2010), 126 (2011), 132 (2012), 142 (2012), 153 (2014, dieser Band), 164 (2015) und dazu noch weitere in anderen Buchreihen.

C. Elsholtz (Graz)

A. Zangwill: Modern Electrodynamics. with 515 b/w illus. 579 exercises, Cambridge University Press, 2013, xxi+977 S. ISBN 978-0-521-89697-9 H/b £ 50.– $ 85,–.

Beim Verfassen seines Lehrbuchs folgte der Autor dem Grundsatz, dass ein Lehrbuch alles enthalten sollte, was die Studierenden wissen müssen, nicht aber alles, was der Autor weiß. Diesem Grundsatz und dem Anspruch folgend, ein thematisch möglichst umfassendes Lehrbuch zu schreiben, entstand ein fast 1000 Seiten starkes Werk, dessen Inhalt an dieser Stelle nur angedeutet werden kann. Nach einer kurzen (aber tiefgehenden) mathematischen Einführung werden die Maxwellschen Gleichungen, die Elektrostatik, dielektrische und leitende Materie sowie Laplace- und Poissongleichung behandelt (Kapitel 1 bis 9). Die nächsten fünf umfangreichen Kapitel sind dem Magnetismus gewidmet. Kapitel 15 bis 21 behandeln die Ausbreitung und Eigenschaften von elektromagnetischen Wellen mit Anwendung der vorgestellten Theorien auf Wechselwirkungen elektromagnetischer Strahlung mit Materie, inklusive einer kurzen Darstellung von Beugungsproblemen. Die letzten drei Kapitel des Buchs schließlich sind der speziellen Relativitätstheorie, den Feldern bewegter Ladungsträger und einer kompakten Darstellung der Lagrange- und Hamiltonformulierung elektrodynamischer Phänomene gewidmet.

Das Buch ist ein Meisterwerk der Didaktik und dies ohne Abstriche in Bezug auf das Niveau der Präsentation des Stoffs. Der Selbstanspruch des Autors ist damit vollkommen erfüllt. Nicht nur die überwältigende Fülle und Tiefe der Themen, sondern auch ihre durchgängig überaus klare und präzise Darstellung machen dieses Buch zu einer sehr wertvollen Lernhilfe für Studierende höherer Semester und zu einer beinahe unerschöpflichen Stoffquelle für Lehrende.

E. Werner (München)

Neue Mitglieder

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