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Inhalt

Gerhard Larcher, Michael Drmota, Peter Grabner, Peter Hellekalek, Roswitha Hofer, Peter Kritzer, Gunther Leobacher, Friedrich Pillichshammer, Robert Tichy, Arne Winterhof: The FWF-Special Research Area "Quasi-Monte Carlo Methods: Theory and Applications" 1

Henk Bruin: The work of Y. Sinaĭ 21

Markus Haltmeier and Sergiy Pereverzyev Jr. : Introduction to the mathematics of computed tomography 29

Buchbesprechungen 53

Nachrichten der Österreichischen Mathematischen Gesellschaft 65

Erratum 68

Neue Mitglieder 69

Dodekaederstern — Titelseite: Vor dem neuen Gebäude der Fakultät für Mathematik der Universität Wien am Oskar Morgenstern-Platz 1 (Roßauerlände) wurde am 28.11.2013 eine Skulptur einer algebraischen Fläche enthüllt, die dieselbe Symmetrie wie dasjenige reguläre Dodekaeder aufweist, in dessen Ecken die 20 Singularitäten der Fläche liegen. Die Gleichung „$f = 0$“ dieser Fläche wurde in der Forschungsgruppe von H. Hauser im Rahmen einer Diplomarbeit bestimmt. Mit der Symmetriegruppe $G \cong A_5 \times \mathbb{Z}_2$ des Dodekaeders ergibt sich das G-invariante Polynom f als eine geeignete Kombination von bekannten Erzeugern u, v, w von $\mathbb{R}[x,y,z]^G$:

$$u = x^2 + y^2 + z^2, \;\; v = (x^2 - \varphi^2 y^2)(y^2 - \varphi^2 z^2)(z^2 - \varphi^2 x^2),$$
$$f = 5c(2\varphi - 3)v - (1-u)^3 + \frac{5}{27}cu^3 \qquad (\varphi = \tfrac{1+\sqrt{5}}{2},\ c = 81).$$

Für mathematische Details siehe: Alexandra Fritz und Herwig Hauser, Platonic Stars. *Math. Intelligencer* 32 (2010), 22–36. Hintergrundinformation zur Skulptur an der Universität Wien finden sich auf *http://www.dodekaederstern.cc*.

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The FWF-Special Research Area "Quasi-Monte Carlo Methods: Theory and Applications"

Gerhard Larcher, Michael Drmota, Peter Grabner, Peter Hellekalek, Roswitha Hofer, Peter Kritzer, Gunther Leobacher, Friedrich Pillichshammer, Robert Tichy, Arne Winterhof

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and Austrian Academy of Sciences

1 Introduction

In December 2013 the Austrian science fund (FWF) granted a special research area (SFB) on the topic "Quasi-Monte Carlo Methods: Theory and Applications". This SFB – which is intended for two four-year periods and which started work in February 2014 – is coordinated by Gerhard Larcher (speaker) and Friedrich Pillichshammer (co-speaker), both from the Johannes Kepler University Linz. It connects ten research projects, led by Michael Drmota (TU Vienna), Peter Grabner and Robert Tichy (both TU Graz), Peter Hellekalek (Paris Lodron University Salzburg), Roswitha Hofer, Peter Kritzer, Gerhard Larcher, Gunther Leobacher, Friedrich Pillichshammer (all Johannes Kepler University Linz), and by Arne Winterhof (RICAM, Austrian Academy of Sciences). The SFB funds make it possible to finance about 20 new Postdoc and PhD positions.

The work in this research project will be accompanied and monitored by an international advisory board of highly renowned experts in Quasi-Monte Carlo (QMC) Methods. The chair of the advisory board is Harald Niederreiter, who is a central

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figure in the field of QMC methods. In his research, he has frequently cooperated with the project leaders for many years, and he will thus play a central role in this SFB.

There is a variety of "big open problems" in QMC, problems partly arising from theory, partly arising from applications. It is the aim of this SFB to efficiently exchange the skills of the participating research groups, to analyze the new modern techniques in QMC and integrate them into the joint work, to develop powerful new methods and so to contribute in an essential way to solutions of the most challenging problems in the field. Further it will create a center of excellence for the theory and the application of QMC methods to be visible worldwide.

"Quasi-Monte Carlo Methods" include all methods in which most carefully chosen quasi-random-point sets are used to carry out simulations in the framework of sophisticated and highly developed modeling environments, for obtaining quantitative information in different branches of applications. The study and development of QMC methods requires

- the generation, investigation, and analysis of distribution properties of finite or infinite sequences in all kinds of regions;
- the development, investigation, and analysis of suitable theoretical models on which the applications of the QMC methods are based, and in particular the derivation of error bounds for QMC methods in these models;
- the efficient implementation of the theoretical models and of the algorithms for the generation of the (sometimes very large and high-dimensional) quasi-random point sets, and the development of sophisticated software;
- the concrete application of the QMC methods in different areas, the discussion of the implications and of the performance of the applied QMC methods.

Consequently, many different branches of mathematics are involved in the comprehensive investigation and development of QMC methods, most notably number theory, discrete mathematics, combinatorics, harmonic analysis, functional analysis, stochastics, complexity theory, theory of algorithms, and numerical analysis. Furthermore, profound knowledge of the branches of applications in which the QMC methods are intended to be used is necessary. The theory and application of QMC methods is a modern and extremely lively branch of mathematics. This is demonstrated by an enormous output of research papers on this topic over the last decades, and by the great and growing success of the series of the biannual international conferences on "Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing" (MCQMC), which started in 1994 in Las Vegas and was most recently held in Sydney in 2012 and in Leuven (Belgium) 2014.

It is the aim of this article to give a short insight into some of the most relevant topics in QMC which will be investigated by the research groups participating in

this SFB. In Section 2 we give a very brief introduction to the basic facts on and techniques used in QMC. In the remaining Sections 3–12 we give examples of some of the main concrete research topics studied in the SFB.

2 Quasi-Monte Carlo Methods: Basic facts and techniques

Many quantitative problems in various fields of applications (e.g., finance, engineering, economics, physics, medicine, biology, ...) involve the task of approximately evaluating (sometimes very high dimensional) integrals. This is particularly often the case when one has to calculate the expected value or the variance of a random variable whose value depends on many random sources.

The basic (quasi-) Monte Carlo approach to evaluate such integrals (say of a function f over an s-dimensional unit cube $[0,1]^s$) is, to choose N points $x_1,\dots,x_N$ in $[0,1)^s$ and to approximate the integral by the average value of f at these sample points, i.e.,

$$\int_{[0,1]^s} f(x)\,\mathrm{d}x \approx \frac{1}{N}\sum_{n=1}^{N} f(x_n).$$

In the pure Monte Carlo approach the N sample points are chosen (pseudo-)randomly. In this case the expected [!] error (i.e., the difference between the true integral value and the approximation) is essentially given by a constant depending on f times $1/\sqrt{N}$.

In QMC methods the sample point sets are chosen deterministically such that the point sets show certain well-distribution properties, and sometimes further structural properties, depending on the class of integrands we are dealing with. In this case the basic error estimate is the fundamental Koksma-Hlawka inequality (see, for example, [13, 18, 27, 30, 36]):

$$\left|\int_{[0,1]^s} f(x)\,\mathrm{d}x - \frac{1}{N}\sum_{i=1}^{N} f(x_i)\right| \le V(f)D_N^*(\{x_1,\dots,x_N\}),$$

where $V(f)$ denotes the variation of f (in the sense of Hardy and Krause) and $D_N^*(\{x_1,\dots,x_N\})$ denotes the star-discrepancy of the point set $\{x_1,\dots,x_N\}$. The star-discrepancy is defined as

$$D_N^*(\{x_1,\dots,x_N\}) = \sup_B \left|\frac{A_N(B)}{N} - \lambda(B)\right|, \tag{1}$$

where the supremum is taken over all axis-parallel boxes B in $[0,1)^s$ anchored at the origin (i.e., which are of the form $B=\prod_{j=1}^{s}[0,t_j)$), where by $A_N(B)$ we denote

the number of indices $n \in \{1,\dots,N\}$ for which x_n is contained in B, and where λ is the s-dimensional Lebesgue measure.

For an infinite sequence $\mathcal{S} = (x_1, x_2, \dots)$ of points in $[0,1)^s$ we denote by $D_N^*(\mathcal{S})$ the star-discrepancy of the point set consisting of the first N elements of the sequence. The sequence is called uniformly distributed, if and only if, $\lim_{N\to\infty} D_N^*(\mathcal{S}) = 0$.

So obviously one of the main tasks in the theory of QMC methods is to analyze the discrepancy of point sets and point sequences, and to provide point sets or point sequences with low discrepancy in a (sometimes very high-dimensional) unit-cube. These tasks – which often lead to deep problems in fields like number theory or combinatorics – are in the center of interest of the SFB and in particular of the projects which will be described in short in Sections 3, 5, 6, 8, 10 and 11 below. It is known that in every dimension s and for all N there exist point sets $\{x_1,\dots,x_N\} \subseteq [0,1)^s$ with star-discrepancy $D_N^*(\{x_1,\dots,x_N\}) \ll_s (\log N)^{s-1}/N$.

Depending on the class of functions one is dealing with the particular integration problem. However, sometimes not only distribution properties of the point sets, but also further structural properties may play a crucial role. To give but one example: Assume that we know that the integrand f is periodic with period one in each coordinate and that all its partial mixed derivatives up to order α exist and are continuous. Then it can be shown that it is of advantage to use so-called good-lattice point sets for numerical integration. These are point sets of the form

$$x_n = \left(\left\{ n\frac{a_1}{N} \right\}, \dots, \left\{ n\frac{a_s}{N} \right\} \right) \quad \text{with } n = 0,1,\dots,N-1, \tag{2}$$

with given integers $a_1,\dots,a_s$. QMC algorithms based on good-lattice point sets are also known as lattice rules, and they were introduced independently by Hlawka and Korobov by the end of the 1950s.

It is known, that for all dimensions s and all N there exist $a_1,\dots,a_s \in \{1,\dots,N\}$, such that the integration error for functions of the above form is of order $O((\log N)^{s\alpha}/N^{\alpha})$.

This is just one classical and well known result in this direction, and – of course – there exists a magnitude of much more subtle integration rules in the modern theory of QMC methods. With the analysis and the development of such efficient integration (and also approximation) rules especially the projects described in Sections 4, 7, 8, 9 and 10 will be concerned.

Finally, the application of QMC methods in concrete problems in most cases needs a suitable adaptation of the methods to the problem. For example, the integration region might not be a unit cube but a more general manifold (e.g., the sphere), or certain variance and variation reduction methods might have to be applied, or the special simulation problem needs point sets with additional pseudo-random properties. With such problems especially the projects described in Sections 4, 9, 11 and 12 will be concerned. When dealing with concrete applications in this SFB,

then in almost all cases we will work with showcase problems from mathematical finance.

In the following we highlight some of the main topics of our research in the SFB.

3 Subsequences of automatic sequences and uniform distribution

This project part is led by Michael Drmota and it aims at constructing uniformly distributed sequences with the help of proper subsequences of automatic sequences.

Automatic sequences are sequences $t(n)$ on a finite alphabet that are the output of a finite automaton. The Thue-Morse sequence $T(n) = s_2(n) \bmod 2$ is one of the most prominent examples of an automatic sequence. (Here and in what follows $s_q(n)$ denotes the q-adic sum-of-digits function).

One of the main motivations for the research in this project part is the recent progress on the so-called Gelfond problems [19] on the prime values and on polynomial values of the sum-of-digits function modulo m. Gelfond conjectured that the subsequence $s_q(p)$, where p runs through all primes, as well as subsequences of the form $s_q(P(n))$, where P is a polynomial of degree greater than 1, are uniformly distributed on the residue classes modulo m. (The cases of primes and squares of these 40 year old conjectures have been solved by Mauduit and Rivat [32, 33], and there is a partial solution for polynomials by Drmota, Mauduit and Rivat [17]). Furthermore, Drmota, Mauduit and Rivat [16] recently proved that the subsequence $T(n^2) = s_2(n^2) \bmod 2$ of the Thue-Morse sequence is actually a normal sequence, that is, every possible 0-1-block appears with the correct asymptotic frequency. Consequently this sequence can be used to generate a Quasi-Monte Carlo sequence. Since automatic sequences (like the Thue-Morse sequence) can be efficiently generated this gives rise to a completely new efficient construction of Quasi-Monte Carlo sequences.

Therefore the *first overall goal* of this subproject is to provide a more systematic treatment to these kinds of problems and to characterize the distributional behaviour of subsequences of automatic sequences $t(n)$ of the form $t(P(n))$ for polynomials P of degree greater than 1, $t(\lfloor n^c \rfloor)$ (for $c > 1$), and $t(p)$ for primes p. It is certainly too ambitious to expect a complete solution in the general case, nevertheless we will work on (at least) the following questions: to study $t(n^2)$, to study $t(\lfloor n^c \rfloor)$ for specific non-integer $c > 1$, to improve results on $s_q(P(n))$ for polynomials, and to study $t(p)$ for special (e.g. invertible) automatic sequences.

The *second overall goal* of this subproject is to study similar questions for more general digital expansions like the Zeckendorff expansion that is based on Fibonacci numbers. The Zeckendorff sum-of-digits funcion $s_Z(n)$ is the (minimal)

number of Fibonacci numbers that are needed to represent n. It is well known that $s_Z(n)$ is uniformly distributed modulo m and that $\alpha s_Z(n)$ is uniformly distributed modulo 1 for irrational α. However, nothing is known on $s_Z(n^2)$ or $s_Z(p)$. It should be mentioned that $s_Z(n) \bmod m$ is not an automatic sequence, nevertheless it is expected that $s_Z(n)$ has similar distributional properties as $s_q(n)$, also regarding its subsequences.

4 Distributing points on spheres and manifolds: minimal energy and designs

This project part led by Peter Grabner aims for a more detailed investigation of point sets of minimal energy and spherical designs. Especially, estimates for the discrepancy of such point sets are of interest; quantifying the distribution properties of such point sets is necessary for applying them to numerical integration.

Minimal energy point sets. For a given compact manifold $M \subset \mathbb{R}^{d+1}$ and a set of N distinct points $X_N = \{x_1, \ldots, x_N\} \subset M$, the Riesz s-energy is defined as $E_s(X_N) = \sum_{i \neq j} \|x_i - x_j\|^{-s}$. A configuration X_N, which minimizes E_s among all N-point configurations, is called a minimal energy configuration. Several questions are of interest in this context:

- the asymptotic behavior of the minimal energy for $N \to \infty$
- the (weak-*) limiting distribution of the measures $\nu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}$
- the discrepancy between these discrete measures and the limiting measure.

One motivation for studying this question is quite classical: how do N mutually repelling particles distribute on a surface?

- For $s = 1$, $d = 2$ these are particles under a Coulomb potential on a surface.
- For $s \to \infty$ this optimization problem becomes the problem of best packing (cf. [8]).
- The resulting point distributions for moderately large N occur in biology (optimal phyllotaxis, viral morphology).

The case $s < \dim(M)$ can be studied by methods from classical potential theory (cf. [29]). The distribution of minimal energy point sets approaches the equilibrium measure. For $s \geq \dim(M)$ the situation changes completely. The corresponding energy integral diverges for all probability measures. Techniques from geometric measure theory could be applied in [21] to show that the limiting distribution $\mu_M^{(s)}$ of the minimal energy distributions is normalized $\dim(M)$-dimensional Hausdorff measure on M, if M is rectifiable.

In [21] it was shown that for $s > d$ the minimal energy of an N point subset X_N on a d-dimensional rectifiable manifold behaves like

$$\frac{C_{s,d}}{H_d(M)^{s/d}} N^{1+s/d},$$

where H_d denotes the d-dimensional Hausdorff measure. For s tending to infinity, $C_{s,d}^{1/s}$ has a limit that is related to the best-packing constant.

Spherical designs. A spherical t-design is a finite set of points $X \subset S^d$ such that

$$\frac{1}{\#X} \sum_{x \in X} p(x) = \int_{S^d} f(x)\, d\sigma(x)$$

for all polynomials p of degree $\leq t$, where σ denotes the normalized surface measure on S^d (cf. [10]). In [10] a lower bound of order t^d could be given, which was shown to be only attained for small values of t. Only recently, it could be shown that $O(t^d)$ points suffice to obtain a t-design (cf. [4]).

5 Arithmetic primitives for uniform distribution modulo 1

The setting underlying this subproject led by Peter Hellekalek is the following. We are given three mathematical objects: X, ω, and f, where X is a nonempty set, $\omega = (x_n)_{n \geq 0}$ is a sequence in X, and $f : X \to \mathbb{C}$ is a function on X. Suppose that X and f are such that $I(f) = \int_X f$ is defined. It is a fundamental property of any notion of uniformly distributed (u.d.) sequences in X that, for a given u.d. sequence ω, the *sample means* $S_N(f, \omega) = (1/N) \sum_{n=0}^{N-1} f(x_n)$ converge to the *expectation* $I(f)$, if the sample size N increases to infinity, for all functions f in a suitable function class $\mathcal{F}$ defined on X.

The above notions call for an appropriate structure on X. Integration requires a measure space structure on X. The concept of u.d. sequences in X leads to the need for construction methods for such sequences, which, in their turn, demand arithmetics on X. If we also want to use some kind of harmonic analysis to study the difference between $S_N(f, \omega)$ and $I(f)$, a short study of [25] will convince the reader that a *(compact abelian) topological group* X is a suitable mathematical environment.

In this subproject of the SFB, we start our research on the s-dimensional torus $(\mathbb{R}/\mathbb{Z})^s$, which we represent by the compact abelian group $X = [0,1)^s$ with addition modulo one. For a given sequence ω in $[0,1)^s$, it is clearly important to measure the uniform distribution of ω. The best known *figures of merit* employed for this

task are *discrepancy* and *diaphony* (see [18, 27, 36]). During the recent years, many other figures of merit for u.d. sequences have been developed and relations to extremal integration errors in certain function spaces have been established (see, for example, [12, 13]).

The first goal of this project is to find a unifying general concept for these figures of merit. We will study a generalized version of the *spectral test*, which is a concept based on so-called *convergence determining classes* of functions. Examples are the trigonometric functions and the Walsh functions. The methods and results developed in [9, 22, 23, 24] will serve as starting points.

The second goal concerns the construction of finite and infinite sequences on the s-torus with good uniform distribution behavior. We will employ the arithmetical structure of the b-adic numbers $\mathbb{Q}_b$ in a new construction method that is related to the method of *good lattice points* (see [36, 46]) and to some duality principles (see [37, 45]).

The third goal is about *arithmetic primitives*. In the construction principles behind cryptographic primitives and behind pseudo-random number generators, the iteration of a given *update function* $f : S \to S$ on a *finite* state space S plays a central role. We are interested in the question of how to describe the long-term behavior of the orbits $x, f(x), f^2(x) = f(f(x)), f^3(x), \ldots$ of a given point $x \in S$ in dependence of certain properties of f. What are the appropriate mathematical models to rate different update functions with respect to their (bit-) mixing behavior? Due to the finiteness of S, there is no asymptotics. Two different approaches to this kind of question can be found in the survey papers [26, 44].

6 Finite-row digital sequences and related hybrid sequences

One main aim of this project part (which is led by Roswitha Hofer) is to deepen the study of hybrid sequences with at least one digital component-sequence. Hybrid sequences are built by concatenating the components of two or more different types of low-discrepancy sequences or in the original idea of Spanier [48] by combining low discrepancy sequences with pseudo-random sequences. The intentions are multiple: combining the different structures and/or advantages of the component sequences, providing new types of sequences, discovering new types of low-discrepancy sequences, etc. The difficulty we face when studying the distribution of hybrid sequences is to work out proper methods which can handle the different structures of the component sequences. Hybrid sequences with one or more digital component sequences appear as particularly hard to study. Digital sequences are constructed by the digital method introduced in [35]. The digital method is an algorithm that generates the n-th point of the s-dimensional sequence

$(x_n)_{n\geq 0}$ by operating on the digits of n in base q and at whose heart are s doubly infinite generating matrices. It should be emphasized here that the distribution of the sequence is mainly determined by the specific choice of the generating matrices, and the main computational effort of the algorithm lies in the multiplication with those matrices. In particular, for hybrid sequences so called finite-row digital sequences, which are generated by matrices satisfying that each matrix row contains only finitely many nonzero entries, seem to be promising.

This project part contains partial problems which are relevant for the investigation of hybrid sequences with at least one digital component-sequence and which are interesting as number-theoretical problems per se. An interesting problem is to determine specific relations between special generating matrices, which for example yield certain correlations between the components of the generated digital sequence. This problem seems to be related to combinatorial objects such as binomial type sequences of polynomials and generalized versions. Thereof such relations between generating matrices are not only interesting for the investigation of hybrid sequences but may also be interesting for efficient construction algorithms of digital sequences. Furthermore, the current methods for investigating hybrid sequences need information on the distribution of specific subsequences of the component sequences. As a part of this project we want to deepen recent investigations of subsequences of digital sequences.

7 Approximation of integrals and functions by new types of quasi-Monte Carlo algorithms

In this project part, led by Peter Kritzer, we consider recent trends in the theory of QMC algorithms applied to problems of integration and approximation over suitably chosen function spaces. A particular emphasis is laid on high-dimensional problems where it is necessary to control how the error of an algorithm depends on the dimension of the problem.

A prominent topic in this project part is that of function approximation by means of QMC (and, more generally, linear) algorithms. The basic problem is to study classes of functions defined on a domain $D \subseteq \mathbb{R}^s$ which usually can be represented by an expansion of the form

$$f(x) = \sum_k \widehat{f}(k) e_k(x),$$

where the e_k form an orthonormal function system and where the coefficients $\widehat{f}(k)$ are given by $\widehat{f}(k) = \int_D f(x)\overline{e_k(x)}\mathrm{d}x$.

The approximation algorithms considered in this project frequently work as follows. We first choose a finite set $\mathcal{A}$ of indices k corresponding to the typically

large coefficients $\widehat{f}(k)$ of the functions considered. Then, the coefficients $\widehat{f}(k)$ are approximated by a suitably chosen QMC algorithm $Q_{N,s}(f,k)$ using N integration nodes. That is, we approximate f by

$$A_{N,s}(f)(x) := \sum_{k \in \mathcal{A}} Q_{N,s}(f,k) e_k(x),$$

which makes it necessary to control both the error of a QMC integration rule and the error of truncating the series expansion of f. The error of an approximation algorithm $A_{N,s}$ is measured in, most notably, the L_2 or L_∞ norm. If the function class we consider is a normed space, we frequently use the so-called worst-case error (i.e., the supremum of the error over the unit ball of the space) as a quality criterion for approximation algorithms.

In our error analysis, we study how the error of an approximation algorithm depends on two quantities: the number N of integration nodes used in the QMC algorithm for approximating the coefficients $\widehat{f}(k)$, and the dimension s of the problem. It is crucial to also include the dependence on the dimension, as our algorithms should ideally work for high-dimensional problems, and we would like to avoid a curse of dimensionality, i.e., we would like to achieve an approximation error that does not depend exponentially on s. If the latter situation occurs, we say that we can achieve tractability, a concept that has been introduced by Woźniakowski in [52]. As outlined in the seminal paper [47] by Sloan and Woźniakowski, one can achieve tractability of multivariate algorithms in certain weighted function spaces, where the influence of different groups of variables is modeled by weights.

So far, there have been numerous results on function approximation based on QMC or related algorithms for functions in certain weighted reproducing kernel Hilbert spaces, as for example in [11] and [28]. In these and in related papers, functions defined on the s-dimensional unit cube $[0,1]^s$ that can be represented as Fourier or Walsh series are considered.

For many of the previous results, one had to make rather restrictive assumptions on the function classes considered, such as smoothness or periodicity assumptions. It is one of the main goals of this project to develop approximation algorithms that also work for more general function classes and to relax some of the restrictions we had to make until now. As first examples, it is intended to study cosine spaces of non-periodic functions defined on $[0,1]^s$ and Hermite spaces of functions defined on $\mathbb{R}^s$.

In all problems of high-dimensional integration and approximation considered in this project, it is our goal to provide constructive algorithms.

8 Improved discrepancy estimates for various classes of sequences

The aim of this project part, which is led by Gerhard Larcher, is to give improved discrepancy estimates for several types of point sequences in an s-dimensional unit cube, but also to give general discrepancy estimates for whole classes of sequences.

It is the so-called "big open problem" in the theory of uniform distribution to determine the best possible order for the discrepancy of point sets in an s-dimensional unit cube. As already mentioned in Section 2, it is known that in every dimension s and for all N there exist point sets with discrepancy $D_N^* \ll_s (\log N)^{s-1}/N$, and that in every dimension s there exist infinite point sequences with discrepancy $D_N^* \ll_s (\log N)^s/N$ for all N. Let us concentrate on infinite point sequences in the following. Examples for such sequences are Halton sequences, or digital (t,s)-sequences in the sense of Niederreiter.

However, for $s \geq 2$ it is not known until today whether the order $D_N^* \ll_s \frac{(\log N)^s}{N}$ for the discrepancy of infinite sequences in the s-dimensional unit cube is the best possible order or not (for $s = 1$ it was shown by W.M. Schmidt in 1972, that the order is best possible). The corresponding best lower bound for $s \geq 2$ currently known was given by Bilyk, Lacey and Vagharshakyan in [3]: There are positive constants c_s and δ_s such that for every sequence $\mathcal{S}$ in $[0,1)^s$ we have

$$D_N^*(\mathcal{S}) > c_s \frac{(\log N)^{s/2+\delta_s}}{N}$$

for infinitely many $N \in \mathbb{N}$. Here δ_s is a positive, but very small constant which goes to 0 for s tending to infinity.

Indeed, until now even for seemingly very simple types of two-dimensional sequences the correct order of discrepancy is not known. A basic example for such a sequence is the simple 2-dimensional Halton sequence in bases 2 and 3.

The Halton sequence is defined as follows: For a nonnegative integer n and an integer $b \geq 2$ let $n = n_r b^r + n_{r-1} b^{r-1} + \cdots + n_1 b + n_0$ be the base b digit representation of n. Define the radical inverse function ϕ_b by

$$\phi_b(n) := \frac{n_0}{b} + \frac{n_1}{b^2} + \cdots + \frac{n_r}{b^{r+1}}.$$

Then the 2-dimensional Halton sequence in bases 2 and 3 is given by

$$x_n = (\phi_2(n), \phi_3(n)) \quad \text{for } n = 0,1,2,\ldots.$$

A further example is the really simple hybrid sequence

$$x_n = (\phi_2(n), \{n\sqrt{2}\}) \quad \text{for } n = 0,1,2,\ldots.$$

For both of these two simple sequences we do not know the correct order of their discrepancy.

It is the main aim of this project part to improve – and in the best case to find the correct order of – the upper and lower discrepancy bounds of frequently used sequences like Halton sequences, digital (t,s)-sequences in the sense of Niederreiter, and of certain types of hybrid sequences.

9 Adapting QMC algorithms to the simulation problem

The project, which is led by Gunther Leobacher, is located at the interface between QMC methods and applications in finance and natural sciences. Hereby the main questions are how to (re-)formulate a given high-dimensional integration problem to make it more suitable for QMC.

One of the most fruitful approaches known is to express the problem as an expectation of a function depending on independent standard normal variables and concatenate the function with a carefully chosen orthogonal transform. Well known examples of general purpose transforms are provided by the Brownian bridge construction or the principal component analysis construction (PCA)construction of Brownian paths. More specialized orthogonal transforms, which take the form of the integrand into account, exist as well. For very high-dimensional problems another important requirement is that the transform can be computed sufficiently fast, whereby the benchmark is the complexity of the PCA construction for Brownian paths, with computational cost $O(n\log(n))$ for an n-dimensional problem. We call this problem of finding a fast efficient orthogonal transform "FEOT problem".

It is a curious fact that the choice of any orthogonal transform does not make a difference for classical Monte Carlo, since for a standard normal vector X and an orthogonal transform U we have $\mathrm{E}(f(X)) = E(f(UX))$. On the other hand QMC algorithms, originally developed for problems of moderate dimension, become more efficient if the problem can be formulated in a way such that the integrand depends mainly on only few of the input parameters while the others have little influence. And frequently this can be facilitated by simply applying an orthogonal transform.

So we may consider an orthogonal transform U to be effective for the integrand f, if only a couple of input parameters of $f \circ U$ are important. A classical concept for measuring the numbers of important parameters is that of "effective dimension", see [6], which relies on the ANOVA decomposition of f resp. $f \circ U$. Thus U could be considered effective, if the effective dimension of $f \circ U$ is much lower than that of f.

A modern alternative to that concept is provided by weighted norms of reprodu-

cing kernel Hilbert spaces, as introduced in [47]. Here the integration error of f can be bounded by the norm of f through a Koksma-Hlawka type inequality, and thus an orthogonal transform U can be considered to be effective for the integrand f, if the weighted norm of $f \circ U$ is much smaller than that of f.

At the present the project has two main goals: 1. to find and study suitable reproducing kernel Hilbert spaces of functions on the $\mathbb{R}^d$ in which integration is tractable and 2. to find algorithms for the FEOT problem in those spaces. A practical problem occuring is that both effective dimension and weighted norms usually do not depend continuously on the orthogonal transform. Thus we need to find, for example, weighted reproducing kernel Hilbert spaces over the $\mathbb{R}^d$ which are invariant under orthogonal transforms of the $\mathbb{R}^d$. One additional constraint on these spaces is that they should contain interesting functions while at the same time integration should be defined and tractable (in the sense of Section 10).

10 Digital nets and lattice based integration rules

In this project, led by Friedrich Pillichshammer, we analyze QMC rules based on lattice point sets in the sense of Hlawka and Korobov (see (2)) and on digital nets and sequences in the sense of Niederreiter [35]. As the quality criterion we study the worst-case integration error of QMC rules in various function spaces, a concept which comprises the notions of classical and weighted discrepancy. We aim at finding explicit constructions of "good" point sets and sequences and we want to study the dependence of the worst-case error on the dimension of the problem. The following two topics are exemplary:

Extending Roth's general lower bound for the L_2 discrepancy of finite point sets from [43], Proinov [41] showed in 1986 that for any infinite sequence $\mathcal{S}$ in $[0,1)^s$ the L_p discrepancy[1] with $p \in (1,\infty)$ satisfies

$$L_{p,N}(\mathcal{S}) \geq c_{s,p} \frac{(\log N)^{s/2}}{N} \quad \text{for infinitely many } N \in \mathbb{N}. \tag{3}$$

Recently, together with Dick [14], we found first explicit constructions of infinite digital sequences over the finite field $\mathbb{F}_2$ with L_2 discrepancy of exactly this order of magnitude, which shows that Proinov's lower bound is best possible for $p \in (1,2)$. For arbitary $p > 2$ this problem is still open, and it is one aim of this project part to find explicit constructions of infinite sequences whose L_p discrepancy matches the lower bound (3). (We remark that for *finite* point sets the problem has already been solved by Chen and Skriganov [7] for $p = 2$ and by Skriganov [45] for arbitrary $p \in (1,\infty)$.)

[1] The star-discrepancy D_N^* given in (1) can be viewed as the L_∞ norm of the local discrepancy $A_N(B)/N - \lambda(B)$. In this sense, the L_p discrepancy is the L_p norm of the local discrepancy.

Classical theories study the dependence of the integration error of QMC rules on the number N of underlying integration nodes. Depending on the smoothness of the integrands, described by a certain parameter α, one can typically achieve an error convergence of the form $O((\log N)^{\kappa_{s,\alpha}}/N^{\alpha})$ for finite smoothness or even $O(\mathrm{e}^{-c_s N^{B_s}})$ for infinite smoothness. E.g., for functions on $[0,1]^s$ with finite mixed partial derivatives up to order one, the worst-case integration error is related to the star-discrepancy of the integration nodes which can be of order $O((\log N)^{s-1}/N)$. Such convergence rates are excellent in an asymptotic sense when N grows to infinity. However, if we still consider the star-discrepancy, the function $N \mapsto (\log N)^{s-1}/N$ is increasing for $N \leq \mathrm{e}^{s-1}$. But already for moderately large dimensions s (e.g., in the hundreds) the value e^{s-1} is too large to use point sets of cardinality $N > \mathrm{e}^{s-1}$ in practical applications. This means that we need to analyze the error bounds of QMC rules also with respect to their dependence on the dimension s. This is systematically done by studying the so-called information complexity $N(\varepsilon, s)$, which is the number of nodes required in order to reduce a certain initial error in dimension s by a factor of ε, where $\varepsilon \in (0,1)$. Problems for which $N(\varepsilon, s)$ grows exponentially in s or ε^{-1} are called intractable, and this is exactly what we want to avoid. If, on the other hand, the information complexity is bounded polynomially in s and ε^{-1}, we speak of polynomial tractability. The subject of tractability for multivariate problems has been introduced by Woźniakowski [52] in 1994. It is a further aim of this project part to study tractability properties for various function spaces and to present explicit constructions of point sets which can achieve tractability. We think that lattice point sets and digital nets and sequences are good candidates for this as well. Following a recent stream of research, we also study the case of infinite smoothness.

11 Diophantine equations, discrepancy and finance

In the analysis of QMC methods, probabilistic methods can be used to investigate the typical behavior of the distribution properties of sequences. An interesting class of sequences, because of its importance in Fourier analysis and in probabilistic number theory, is the class of lacunary sequences $(n_k x)_{k=1}^{\infty}$ for $x \in \mathbb{R}$, where (n_k) is exponentially growing: $n_{k+1}/n_k \geq q > 1$. Answering a question of P. Erdős, Walter Philipp (1975) proved a "bounded" law of the iterated logarithm (LIL) for the discrepancy[2] of such sequences:

$$\frac{1}{8} \leq \limsup \sqrt{\frac{N}{2 \log\log N}} D_N(n_k x) \leq C(q) \tag{4}$$

[2]The discrepancy D_N of a sequence is defined in the same way as the star-discrepancy in (1) with the only difference that the supremum is extended over *all* axes-parallel boxes of the form $B = \prod_{j=1}^{s} [u_j, v_j)$ in $[0,1)^s$.

for almost all x (in the sense of Lebesgue measure on $\mathbb{R}$) with a constant $C(q)$ depending on the growth rate q; see for instance [18]. Note that if (ξ_k) is a sequence of i.i.d random variables on $(0,1)$ then

$$\limsup_{N\to\infty} \sqrt{\frac{N}{2\log\log N}} D_N(\xi_k) = \frac{1}{2} \tag{5}$$

with probability one by Chung-Smirnov LIL. It is one aim of this subproject, led by Robert Tichy, to investigate the "probabilistic" behavior of deterministic sequences. This involves various tools, mainly from Fourier analysis, martingale inequalities and methods from Diophantine analysis such as the theory of S-unit equations. It was for instance shown in papers by C. Aistleitner, I. Berkes and R. Tichy [1, 2] that a LIL with constant $\frac{1}{2}$ as in (2) holds for lacunary sequences $(n_k x)$ provided that $n_{k+1}/n_k = \infty$ ("strongly lacunary sequences"). Furthermore, this result is permutation independent, i.e. it remains true for sequences $(n_{\sigma(k)}x)$, where $\sigma : \mathbb{N} \to \mathbb{N}$ is an arbitrary permutation of the positive integers. In the case of "proper" lacunary sequences i.e. $\limsup_{k\to\infty} \frac{n_{k+1}}{n_k} = q > 1$ the situation is completely different: the constant in the LIL depends on the growth rate q and the result is in general not permutation invariant. Within this project the investigation will be extended to more general classes of sequences, in particular to the multidimensional situation and to certain sublacunary sequences. In this context so-called Hardy-Littlewood-Pólya sequences are well understood because of their arithmetic structure: in this case (n_k) is given as the multiplicative semigroup generated by finitely many coprime integers and arranged in increasing order. Such sequences were used by H. Furstenberg in the theory of dynamical systems. Later W. Philipp (1994) proved a "bounded" LIL for this class of sublacunary sequences. By Diophantine tools C. Aistleitner, I. Berkes and R.F. Tichy obtained a permutation invariant LIL, and it remains open to extend such results to more general classes of sublacunary sequences and to other kinds of distribution measures. It is also one aim of this subproject to apply Diophantine and probabilistic tools to the analysis of models in financial mathematics.

12 On the hierarchy of measures of pseudorandomness

This project part deals with the analysis of pseudorandom numbers in view of several different application areas. It is led by Arne Winterhof.

Pseudorandom numbers are generated by deterministic algorithms and are not random at all. However, in contrast to truly random numbers, they guarantee certain randomness properties. Their desirable features depend on the application area. For example, uniformly distributed sequences of pseudorandom numbers are nee-

ded for Monte Carlo methods, unpredictable sequences for cryptography, and uncorrelated sequences for wireless communication or radar. Some corresponding quality measures are discrepancy for uniform distribution, linear complexity for unpredictability, and autocorrelation.

The main goal of this project is finding relations between different measures of pseudorandomness. For example, the linear complexity provides essentially the same quality measure as certain lattice tests coming from the area of Monte Carlo methods, see [15, 38]. Moreover, the paper [31] studies links between uniformly distributed pseudorandom sequences (x_n) of real numbers in $[0,1)$ and the pseudorandom binary sequences (e_n) defined by $e_n = 0$, if $x_n < 1/2$ and $e_n = 1$ otherwise. It is proved that good pseudorandom $[0,1)$ sequences induce binary sequences that have small correlation measures. The correlation measure of order k is a rather general measure of pseudorandomness introduced by Mauduit and Sárközy [34]. A relation between linear complexity and the correlation measure of order k is given in [5]. Hence, we may very roughly say that discrepancy is a stronger measure than the correlation measure which is a stronger measure than linear complexity. There are many other related measures of pseudorandomness for sequences, see [20, 42, 49], and we want to analyze their hierarchy. In this hierarchy we may also include measures for cryptographic functions. For example, a small correlation measure of order k of a binary sequence guarantees a high nonlinearity and algebraic degree of a corresponding Boolean function [40], which is necessary to avoid some cryptanalytic attacks.

Moreover, we try to find explicit sequence constructions which separate the hierarchy classes and have excellent behaviour under the strongest measures. A focus is put on uniformly distributed sequences derived from dynamical systems, see the survey [50], hybrid sequences, sequences defined using characters of finite fields, and interleaved sequences. We will also study relations to emergent areas as coding theory, biology, or quantum computing.

Our main tools are from analytic number theory, in particular, exponential sum or character sum techniques. For recent surveys on character sums and their applications see [39, 51]. However, we also use very new techniques for example from additive combinatorics.

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The work of Y. Sinaĭ

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On May 19 2014, Professor Yakov Sinaĭ (Moscow State University) was awarded the Abel Prize for his contributions to mathematical physics. We sketch some of his main achievements.

Receiving the Abel prize for Professor Yakov Sinaĭ and his wife Elena Vul (also a mathematician) entailed a varied program of journeys (Oslo, Stavanger and Stockholm), lectures, receptions and interviews (with the Press and Martin Raussen & Christian Skau; the latter traditionally appears in the Notices of the AMS). The highlight was of course the award ceremony itself, at the hands of Crown Prince Haakon of Norway, on May 20 2014.

Yakov Grigorevich Sinaĭ was born in 1935 in a family of scientists. His parents were both microbiologists in Moscow and his grandfather a prominent mathematician, head of department of the differential geometry at Moscow State University. Sinaĭ obtained his first degree in 1957, which was also the year of his first publication. His master (equivalent to PhD) degree followed in 1960. The academic landscape in Moscow, within the rapidly developing fields of Ergodic Theory and Statistical Mechanics was truly remarkable: Chataev, Dynkyn and Kolmogorov were his advisors, and the faculty included Anosov, Krylov, Dobrushin, Gel'fand and others. The fact that he came from a Jewish family, however, restricted his possibilities in the Russian system, and he was unable to get a full position at the Mathematics Department. Instead, he accepted a position at the Landau Institute of Theoretical Physics of the USSR Academy of Sciences. This enabled him to collaborate with physicists as well as mathematicians, and to bridge the two disciplines, as he would continue to do in an unparalleled way. He introduced fundamental concepts of statistical physics into mathematics (Kolmogorov-Sinaĭ entropy, thermodynamic formalism, renormalization groups) giving them a rigorous basis.

Sinaĭ was a crucial figure in spread of ergodic theory. At the time, it was common for talented mathematicians in Eastern Europe to study in Moscow, and this is how Fritz, Krámli and Szász from Budapest and Krzyzewski and Szlenk from Warsaw first came in contact with the emerging field, and in due time founded schools in their home countries.

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Sinaĭ's work was well-known in the West early on, and in 1962 he was invited to give a plenary lecture at the ICM in Stockholm. Russian mathematicians being allowed to travel to the West were more the exception than the rule. For example, the support of the dissident poet-mathematician Esenin resulted in Sinaĭ being barred from giving an address at the ICM in Nice in 1970. He was not alone in this; for example, Novikov was unable to come to Nice and accept the Fields medal in person.

In 1992, Sinaĭ joined the faculty of Princeton University, dividing his time between Moscow and Princeton from that time onwards, in addition to several guest professorships (such as Caltech in 2005). The Abel Prize is currently the last of a long list of awards: the Boltzmann Gold Medal (1986); the Heineman Prize (1989); the Markov Prize (1990); the Dirac Medal (1992); the Wolf Prize in Mathematics (1997); the Brazilian Award of Merits in Sciences (2000); the Moser Prize (2001); the Nemmers Prize in Mathematics (2002); the Henri Poincaré Prize (2009); the Dobrushin International Prize (2009).

Sinaĭ's major work lies in Statistical Mechanics. The aim of this area is to derive the statistical "macroscopic" behaviour of material (which can be gases, liquids, but also fixed atoms or moving electrons in a grid) from the behaviour of the individual particles it consists of. This goes back to James Clerk Maxwell (1831–1879) and Ludwig Boltzmann (1844–1906), who applied the notions of *ergodicity* and *entropy*, although the modern form of these notions (partly due to Sinaĭ) is quite different from Boltzmann's original approach. The Laws of Thermodynamics predict that a system of particles strives to minimal energy and maximal entropy (= disorder), and the work of Josiah Willard Gibbs (1839–1903) united this by the introduction of equilibrium states. These are measures assigning probabilities to configurations of the system, where the weights are inverse proportional to the exponential of the potential energy of the configuration. The fact that particular energy levels are achieved by vastly more configurations than other energy levels, creates an equilibrium between energy and entropy. In principle, the system can move away from the equilibrium, but if the number of particles is large[1], this becomes astronomically unlikely.

In his derivation of the H-Theorem (now called second law of thermodynamics) Boltzmann needed the so-called *Ergodic Hypothesis*:

> *The trajectory of the point representing the state of the system in phase space passes through every point on the constant-energy hypersurface of the phase space.*

This was criticised, not just as mathematically unfeasible, but also it was unknown if any system satisfied this hypothesis, even in a weakened form proposed in the

[1] Which is of course the case in practical situations; in fact the sheer number of particles makes the system completely intractable by deterministic methods.

influential survey paper by Tatiana and Paul Ehrenfest [7]. A rigorous proof for even the simplest systems of a few colliding particles (mathematical billiards) was beyond the state of mathematics at the time, and rigorous definitions of ergodicity (and entropy) had yet to be formulated.

In the 1930s, progress was made in the study of geodesic flow on surfaces of negative curvature. These are non-Euclidean "hyperbolic" surfaces on which initially close trajectories diverge at an exponential rate. This can be considered as a continuous version of dispersion as opposed to the dispersion at discrete time collisions of the particles in the billiard system. The first proofs of ergodicity (Hadamard [8], Artin [1]) for certain hyperbolic geodesic flows relied on number theoretic properties (continued fractions, the Gauß map), but in 1939, Eberhard Hopf (1902–1983) designed a general method of proving ergodicity for hyperbolic flows Φ_t. This became known as *Hopf Chains*, and relies on the fact that ergodic averages $\lim_{T\to\infty}\frac{1}{T}\int_0^T \psi\circ\Phi_t dt$ are constant on stable and unstable sets of points in configuration space. However, this method required smoothness, with stable and unstable sets stretching sufficiently far so as to create net spanning the entire configuration space. This condition is fulfilled for many geodesic flows, but not for systems of colliding particles.

After about 20 years of no progress, the Russian school started to get involved. Using methods from measure theory, Andrej Kolmogorov (1903–1987) and Sinaĭ were able to formalize entropy in a effective way [12, 18]. A measure μ on the a (configuration) space $(X,\mathcal{B})$ is called flow-invariant, if $\mu(A)=\mu(\Phi_t A)$ for all sets $A\subset\mathcal{B}$ and time $t\in\mathbb{R}$. Energy preserving (Hamiltonian) flows have a natural invariant measure, called Liouville measure, but there are many others.

Given a measure μ and a partition $\mathcal{Q}$ of the configuration space, the *entropy* of this partition is given by the sum

$$H(\mathcal{Q},\mu) = -\sum_{Q\in\mathcal{Q}} \mu(Q)\log\mu(Q),$$

and it takes its maximal value $\log\#\mathcal{Q}$ when μ distributes the mass evenly over all partition elements $Q\in\mathcal{Q}$. This reflects that the entropy becomes largest when the probability of finding yourself in a particular state is spread the most. Assuming a discrete-time flow Φ_n for simplicity, let $\mathcal{Q}_n=\vee_{k=0}^{n-1}\Phi_{-k}\mathcal{Q}$ be the n-th joint of the partition; elements in $\mathcal{Q}_n$ are those sets of x that visit the same elements of $\mathcal{Q}$ in time steps $k=0,\dots,n-1$. The Kolmogorov-Sinaĭ entropy is now computed as the growth rate of $H(\mathcal{Q}_n,\mu)$, and then maximized over all finite partitions $\mathcal{Q}$. That is:

$$h_\mu(\Phi_n) = \sup_{\mathcal{Q}} \lim_{n\to\infty}\frac{1}{n}H(\mathcal{Q}_n,\mu).$$

Kolmogorov-Sinaĭ entropy became a widespread tool, also beyond statistical mechanics. There are parallels to Information Theory developed by Shannon [16] in the 1940s; I would also like to mention Ornstein's remarkable theorem [13] that

in the context of two-sided Bernoulli shifts, entropy is a complete isomorphism invariant: two Bernoulli shifts are isomorphic if and only if they have the same entropy.

In the early 1970s, Sinaĭ, combined entropy with the potential energy U of Gibbs' approach to what is known as thermodynamic pressure:[2]

$$P(\beta U) = \sup\{h_\mu(\Phi_t) - \beta \int U d\mu\}.$$

Here the supremum is taken over all flow-invariant measures μ and the parameter $\beta = 1/kT$ for Boltzmann's constant k and absolute temperature T. Those flow-invariant measures μ that achieve this supremum play the role of equilibrium states in Gibbs' approach, in the sense that they realize equilibrium between the (maximal) entropy and (minimal) potential energy. Under some regularity conditions, equilibrium states satisfy the *Gibbs property* which means that the mass of sets $Q \in \mathcal{Q}_n$ scales as the exponential of the ergodic sum of $U - P$, i.e., $\mu(Q) \sim \exp(\sum_{k=0}^{n-1} U \circ \Phi_k - P(\beta U))$. As function of the inverse temperature parameter β, equilibrium states can vary continuously, or abruptly. The latter case is referred to as *phase transition*, in analogy between abrupt chance of equilibrium describing e.g. water in liquid versus frozen form, or a piece of iron in magnetised versus demagnetised form (cf. the Ising model). With this, Sinaĭ [21] laid the foundation for thermodynamic formalism in dynamics. Further contributions come from Rufus Bowen [2] and David Ruelle[3] [14]. A modern text book in ergodic theory with emphasis on this material was written by G. Keller [11].

Coming back to the billiard systems and the Hopf argument, which, as we mentioned, breaks down for billiard systems. Colliding participles can create *singularities* in the flow, when they collide tangentially (grazing collisions) or in three or more at exactly the same time. Such singularities create discontinuities in configuration space and prevent the proper construction of stable and unstable sets, making it impossible to carry out Hopf's argument. It was Sinaĭ who forced the breakthrough by showing that in sense of Liouville measure, stable and unstable sets can be defined and are sufficiently long at "most" points of configuration space.

The basic setup of a billiard flow is a particle (or several particles) moving with constant speed in some region Q (the billiard table) and reflecting elastically against the boundary ∂Q, so that no kinetic energy is lost in collisions, and the angle of incidence is the angle of reflection, see Figure 1. In formula, the velocity v' after collision is identified with the velocity before collision via

$$v' = v - 2\langle v, n(q)\rangle n(q), \qquad q \in \partial Q, \tag{6}$$

[2]The connections with pressure from Newtonian physics is all but lost here.

[3]See [15] and [24] for some mutual 65th birthday wishes between the two.



Figure 1: Collision rule (left) and the billiard flow with one spherical scatterer on the two-torus (right).

where $n(q)$ is the inward pointing normal vector at the collision point q of the boundary of the table, see [25]. The phase space M is the unit tangent bundle of Q, with identifications at the boundary according to (6). The billiard flow Φ_t preserves Liouville measure $dqd\omega_q$, where ω_q is uniform measure on the sphere of unit tangent vectors at $q \in Q$. Rather than the flow Φ_t, we can look at the collision map $F : \partial Q \times S \to \partial Q \times S$, where S is the "half-sphere" obtained from the unit sphere by the identification (6). The map F preserves a measure $\sin\theta d\theta dr$, for $r \in \partial Q$ and angle $\theta \in S$ with the normal vector $n(q)$. For this, ∂Q has to be piecewise smooth; corners of the billiard table, but also grazing collisions with the boundary, give rise to singularities. Usually, the particles are treated as hard spherical objects: the collision of two particles becomes just part of the regular boundary of the billiard table, but simultaneous collisions of three or more particles become "corners" of the billiard.

The first model that Sinaĭ managed to solve this way is a two-particle system on a two-dimensional torus, or equivalently a single particle colliding with a spherical scatterer in the two-dimensional torus, see Figure 1. The general version of this result is known as the Boltzmann-Sinaĭ postulate [19]:

> *The system of N spherical particles with elastic collisions on the d-dimensional torus is ergodic.*

In trying to extend this result to more particles and higher dimensions, addition technicalities come into play: finite versus infinite horizons, semi-dispersing versus fully dispersing billiard, cusps and other intricacies of the geometry. Together with Leonid Bunimovich [4] and later Nikolai Chernov, new techniques were introduced, [5, 4]. Gradually the Hungarian school (Szász, Krámli, Simányi, Bálint, ...) got more involved, also with the help of another Sinaĭ student Dolgopyat. This finally led to the full proof of the Sinaĭ-Boltzmann postulate by Nándor Simányi in 2013 [17]. Current directions in this field try to address the question of (rates of) mixing and further statistical properties of billiard flows. Without doubt, Sinaĭ's work, insights and encouragement over the span of sixty years have carved the landscape of billiard flows like nothing else.

In have restricted my discussion to areas that I am familiar with, leaving out

Sinaĭ's further work on renormalization groups [23], Schrödinger operators [6], fluid mechanics, Navier-Stokes equations (with K. Khanin, J. Mattingly and D. Li), in fact countless topics in mathematical physics, but also number theory and stochastics (e.g. random walks in random environment [22]. His further contributions to dynamical systems include work on Markov partitions for hyperbolic systems (billiards [5, 3]), SRB-measures for (non-uniformly hyperbolic) systems, and there is his expository work of numerous text books, survey articles, and lecture series. Let me finally mention his prominent role in the mathematics community as a whole, and the impressive list of students that he supervised over the years. These include the already mentioned Bunimovich, Chernov, Mattingly and Dolgopyat, but also Bufetov, Gurevich, Jitomirskaya, Katok, Kornfeld, Margulis, Ratner and Ulcigrai.

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Introduction to the mathematics of computed tomography

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1 Introduction

Computed tomography (CT) is one of the most important diagnostic tools in modern medicine. While the term *computed tomography* was initially reserved for x-ray based CT scanners, it nowadays covers various non-invasive imaging technologies, where mathematics plays a major role for obtaining diagnostic images. Examples include x-ray CT, single photon emission computed tomography (SPECT), positron emission tomography (PET), magnetic resonance tomography (MRT), ultrasound tomography, electrical impedance tomography, optical imaging, as well as photoacoustic tomography and the closely related thermoacoustic tomography.

The unifying element of all tomographic applications is that only indirect information about the quantity of interest (usually modelled as a function defined on $\mathbb{R}^2$ or $\mathbb{R}^3$) can be collected when scanning the patient. Due to the modeling imperfections, measurement errors and statistical uncertainties, the data are additionally corrupted by deterministic or random noise. Such type of applications are most conveniently be studied in the framework of inverse problems, where the reconstruction problem is formulated as an operator equation

$$Y = \mathcal{K}f + \varepsilon.$$

Here $\mathcal{K}$ is a linear or nonlinear operator modeling the particular inverse problem, f is the unknown (infinite dimensional) parameter, ε is the noise, and Y are the given noisy data. In many medical imaging technologies the operator $\mathcal{K}$ can be modelled as a Radon transform, which maps a function to its integrals over curves or other manifolds. For example, the data in the classical x-ray CT as well as in

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single photon emission tomography provide approximate integrals of the unknown parameter over straight lines. In the more recent photoacoustic tomography, the underlying transform is the spherical Radon transform, which integrates the unknown function over spherical surfaces.

In this note, we highlight two prime examples of computed tomography, namely classical x-ray based CT and the more recent photoacoustic tomography. For both applications we present the underlying modeling equations. We further discuss basic mathematical results that form the foundation of the filtered backprojection algorithm, which is still the most widely used reconstruction algorithm in medical CT scanners.

2 The classical Radon transform

X-ray CT is the oldest non-invasive medical imaging methodology, where mathematical reconstruction algorithms play a major role for creating slice images of some patient. Like medical radiography, it is based on the physical properties that x-rays mainly propagate along straight lines in tissue and that the spatially varying x-ray attenuation depends on the structure in the interior of the patient. However, in radiology only projection images (averages of the attenuation function over lines) are captured and displayed, whereas x-ray CT uses mathematical reconstruction algorithms combining several projection images to provide section images of the interior of the patient as final output.

The Radon transform, which maps a function defined in the Euclidian plane to its integrals over straight lines, forms the mathematical basis of x-ray CT. Image reconstruction in x-ray CT therefore requires a precise understanding of the Radon transform and in particular requires methods for its analytical or numerical inversion. In 1963, Cormack [9] was the first to point out the possible application of the Radon transform for medical applications. The first commercially available CT system was constructed by Hounsfield [37], and the first patient brain-scan in a hospital was made in 1972. In 1979, Cormack and Hounsfield shared the Nobel Prize for Medicine and Physiology for the development of computed tomography. Later Cormack realized that the transform he studied was already analyzed in 1917 by Johann Radon (see [61]), an Austrian mathematician interested in the problem of recovering a function from its line integrals from a purely mathematical perspective. A long time before the invention of computed tomography Radon already derived an inversion formula for the transform that was later named after him. Radon itself was inspired by work of another Austrian mathematician, Paul Funk, who studied a similar problem, namely that of recovering a function on the two-dimensional sphere from its integrals over all great circles [21].

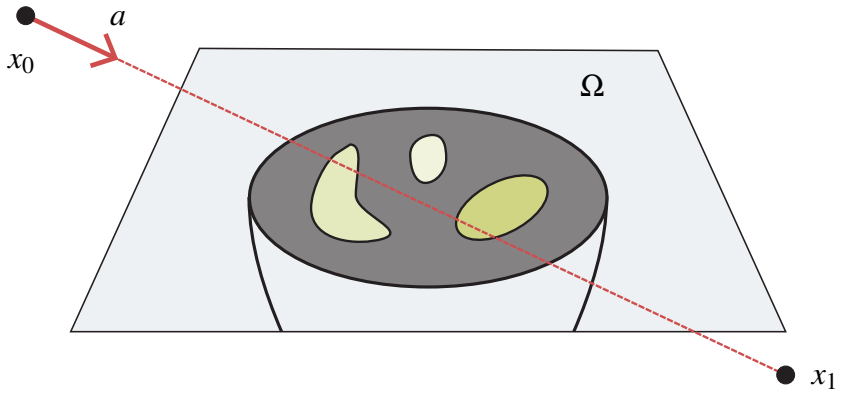


Figure 1: An x-ray is emitted at some point x_0 outside of Ω, propagates along the line L through Ω, and is finally recorded at another location x_1 outside of Ω.

2.1 Mathematical modeling of x-ray CT

For the following, let $\Omega \subset \mathbb{R}^2$ be some convex domain in the Euclidian plane modeling a slice of some human patient. We denote by $f\colon \mathbb{R}^2 \to \mathbb{R}$ the spatially varying x-ray absorption coefficient which is assumed to be supported in Ω. Suppose further, that an x-ray beam origins at some position x_0 outside of Ω, propagates along a straight line L and is finally recorded at another point x_1 outside of Ω; see Figure 1. We parameterize the line L by $\gamma\colon \mathbb{R} \to \mathbb{R}^2\colon t \mapsto x_0 + ta$, where $a \in \mathbb{R}^2$ is a unit vector pointing from x_0 to x_1 and denote by $I(t)$ the intensity of the x-ray beam at location $\gamma(t)$. The intensity at the source position x_0 will be denoted by I_0 and the intensity at the receiver position x_1 by I_1.

According to Beer's law, the loss of intensity in a small interval $[t, t+\Delta t]$ is approximately proportional to the intensity $I(t)$, the attenuation coefficient $f(\gamma(t))$, and the length Δt of the interval. Hence we have $I(t+\Delta t) - I(t) \approx -f\big(\gamma(t)\big)I(t)\Delta t$ and taking the limit $\Delta t \to 0$ yields the initial value problem

$$\begin{cases} \dfrac{dI}{dt}(t) = -f\big(\gamma(t)\big)I(t) & \text{for } t \in \mathbb{R} \\ I(0) = I_0. \end{cases}$$

Integrating this equation gives $I(t) = I_0 \exp\big(-\int_0^t f\big(\gamma(t)\big)\mathrm{d}t\big)$. Evaluating this expression at the special value $t_1 = |x_1 - x_0|$ (corresponding to the detector location) and using $I_1 = I(t_1)$ yields

$$\int_L f(x)\mathrm{d}s(x) := \int_0^{t_1} f(\gamma(t))\mathrm{d}t = \log\left(\frac{I_0}{I_1}\right). \tag{7}$$

From (7) we conclude, that every pair of intensity I_0 emitted by some x-ray source

at x_0 and intensity I_1 measured by an x-ray detector at x_1 provides the integral of f over the straight line through the points x_0 and x_1.

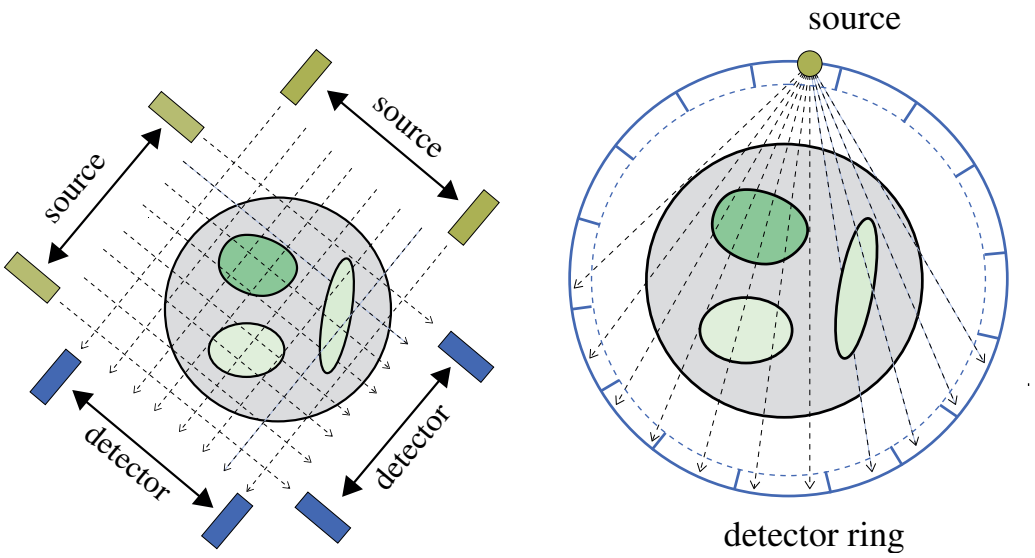


Figure 2: LEFT: In a *first generation CT scanner* a single source and detector pair is translated linearly. Subsequently the source and detector are rotated, and the measurement process is repeated for different orientations. RIGHT: In modern *fourth generation CT scanner* a single source sends out a fan-shaped bunch of x-rays that are recorded with a detector ring surrounding the patient. Subsequently the source is rotated and the measurement process is repeated with different source locations.

By varying the positions of the x-ray sources and detectors, respectively, one collects several integrals of f over different lines. The mathematical task of CT is to recover the function f from these line integrals. The first CT scanner operated in parallel beam mode. As illustrated in the left picture in Figure 2, the source and detector translate linearly, where at any instance a single line integral is collected. Subsequently, the whole apparatus is rotated by a certain angle, and the measurements are repeated until the whole angular range is covered. Such type of scanners are now known as first generation x-ray scanners. Modern fourth generation x-ray scanners operate in fan beam shape where a whole bunch of x-rays is emitted from a single source which rotates around the object of interest (see the right picture in Figure 2).

2.2 The Radon transform

The Radon transform, which integrates a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ over all lines, forms the mathematical basis of x-ray tomography. Let us write any line in the plane in

the form $L = \{s\theta + t\theta^\perp : t \in \mathbb{R}\}$, where $\theta \in S^1$ is a normal vector, $s \in \mathbb{R}$ is the oriented distance of the line from the origin, and $\theta^\perp \in S^1$ denotes a unit vector orthogonal to θ.

Definition 1 (Radon transform)**.** The *Radon transform* $\mathcal{R}f\colon S^1 \times \mathbb{R} \to \mathbb{R}$ of an integrable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$(\mathcal{R}f)(\theta, s) := \int_{\mathbb{R}} f(s\theta + t\theta^\perp)\mathrm{d}t.$$

For fixed $\theta \in S^1$ we call the univariate function $(\mathcal{R}f)(\theta, \cdot)\colon \mathbb{R} \to \mathbb{R}$ the linear projection of f orthogonal to θ.

By Fubini's theorem $(\mathcal{R}f)(\theta, s)$ is well defined for almost any $(\theta, s) \in S^1 \times \mathbb{R}$. We sometimes suppose that f is supported in the open unit disc $D = \{x \in \mathbb{R}^2 : |x| < 1\}$. One easily shows, that $\mathcal{R}$ then defines a linear bounded operator $\mathcal{R}\colon L^2(D) \to L^2(S^1 \times (-1, 1))$, see [48].

Two main theorems

The most basic and probably most important result for the Radon transform is the Fourier slice theorem, that relates the Radon transform to the Fourier transform. For that purpose we denote by

$$(\mathcal{F}f)(\xi) := \int_{\mathbb{R}^d} f(x)e^{-i\langle \xi, x\rangle}\mathrm{d}x \qquad \text{for } \xi \in \mathbb{R}^d,$$

the d-dimensional Fourier transform and by $(\mathcal{F}_2 g)(\theta, \sigma) := (\mathcal{F}g(\theta, \cdot))(\sigma)$ the Fourier transform of a function $g\colon S^1 \times \mathbb{R} \to \mathbb{R}$ in the second argument.

Theorem 2 (Fourier slice theorem)**.** *For any integrable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ we have*

$$(\mathcal{F}f)(\sigma\theta) = (\mathcal{F}_2\mathcal{R}f)(\theta, \sigma) \qquad \textit{for } (\theta, \sigma) \in S^1 \times \mathbb{R}. \tag{8}$$

Proof. This is a simple application of Fubini's theorem. In fact, by Fubini's theorem and the orthonormality of θ and $\theta^\perp$, we have

$$\begin{aligned}(\mathcal{F}_2\mathcal{R}f)(\theta, \sigma) &= \int_{\mathbb{R}} e^{-i\sigma s} \int_{\mathbb{R}} f(s\theta + t\theta^\perp)\mathrm{d}t\mathrm{d}s \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\langle \sigma\theta, s\theta + t\theta^\perp\rangle} f(s\theta + t\theta^\perp)\mathrm{d}t\mathrm{d}s = (\mathcal{F}f)(\sigma\theta),\end{aligned}$$

where the last equality follows by the change of variables $x = s\theta + t\theta^\perp$. □

The argument $\sigma\theta$ appearing on the left hand side of (8) fills in the whole Fourier plane, which is required to invert the Fourier transform using the well known, explicit and stable Fourier inversion formula $f(x) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}(\mathcal{F}f)(\xi)e^{i\langle\xi,x\rangle}\mathrm{d}\xi$. Hence the function f can be reconstructed by means of a one-dimensional Fourier transform, followed by an interpolation based on (8), and finally performing an inverse two-dimensional Fourier transform. Note however, that interpolation in the Fourier domain is a critical issue, and such Fourier domain algorithms have not been very successful in early stages of CT. More recently, such type of algorithms have been improved significantly using ideas from nonuniform fast Fourier transforms [5, 12, 16, 20, 24, 25, 59] or by gridding techniques originally developed for magnetic resonance tomography [53, 67].

While reconstruction algorithms based on the Fourier slice theorem exist, much more common are algorithms of the filtered back-projection type. Such algorithms are based on explicit inversion formulas we shall consider next. For that purpose we denote by

$$(\mathcal{H}_2 g)(\theta,s) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{g(\theta,t)}{s-t} \qquad \text{for } (\theta,s)\in S^1\times\mathbb{R}, \tag{9}$$

the Hilbert transform of a function $g\colon S^1\times\mathbb{R}\to\mathbb{R}$ applied in the second argument. Here the integral is understood in the Cauchy principal value sense. The Hilbert transform is the convolution with the distribution P.V.$[1/s]$ and has the well known Fourier representation $(\mathcal{F}_2\mathcal{H}_2 g)(\theta,\sigma) = -i\,\mathrm{sign}(\sigma)(\mathcal{F}_2 g)(\theta,\sigma)$. Likewise we denote by $\partial_2 g$ the derivative of g in the second argument.

Theorem 3 (Filtered back-projection type inversion formula)**.** *For any continuously differentiable function $f\colon\mathbb{R}^2\to\mathbb{R}$ with support in D, we have*

$$\begin{aligned} f(x) &= \frac{1}{4\pi}\int_{S^1}\left(\mathcal{H}_2\partial_2\mathcal{R}f\right)(\theta,\langle\theta,x\rangle)\,\mathrm{d}\theta \\ &= \frac{1}{4\pi^2}\int_{S^1}\left(\int_{\mathbb{R}}\frac{(\partial_2\mathcal{R}f)(\theta,t)}{\langle\theta,x\rangle-t}\mathrm{d}t\right)\mathrm{d}\theta \qquad \textit{for } x\in\mathbb{R}^2. \end{aligned} \tag{10}$$

There are several different ways to derive the important inversion formula (10); see for example [35, 42, 48]. Below we shall give a simple proof based on the Fourier slice theorem.

Proof of Theorem 3. By the two-dimensional Fourier inversion formula, the use of polar coordinates $\xi = \sigma\theta$, and the Fourier slice theorem we have

$$\begin{aligned} f(x) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}(\mathcal{F}f)(\xi)e^{i\langle\xi,x\rangle}\mathrm{d}x \\ &= \frac{1}{4\pi^2}\int_{S^1}\int_0^\infty \sigma(\mathcal{F}_2\mathcal{R}f)(\theta,\sigma)e^{i\sigma\langle\theta,x\rangle}\mathrm{d}\sigma\mathrm{d}\theta \end{aligned}$$

$$= \frac{1}{8\pi^2} \int_{S^1} \int_{\mathbb{R}} |\sigma| \, (\mathcal{F}_2 \mathcal{R} f)(\theta, \sigma) e^{i\sigma\langle\theta, x\rangle} \mathrm{d}\sigma \mathrm{d}\theta.$$

The Fourier representations of the Hilbert transform and the derivative in the second argument show $|\sigma| \, (\mathcal{F}_2 \mathcal{R} f)(\theta, \sigma) = (\mathcal{F}_2 \mathcal{H}_2 \partial_2 \mathcal{R} f)(\theta, \sigma)$. Hence, application of the one-dimensional Fourier inversion formula yields

$$f(x) = \frac{1}{8\pi^2} \int_{S^1} \int_{\mathbb{R}} (\mathcal{F}_2 \mathcal{H}_2 \partial_2 \mathcal{R} f)(\theta, \sigma) e^{\sigma\langle\theta, x\rangle} \mathrm{d}\sigma \mathrm{d}\theta$$
$$= \frac{1}{4\pi} \int_{S^1} (\mathcal{H}_2 \partial_2 \mathcal{R} f)(\theta, \langle\theta, x\rangle) \mathrm{d}\theta,$$

which is the first claimed inversion formula. The second inversion follows after inserting the definition of the Hilbert transform in the former. □

There exist several other explicit inversion formulas for the Radon transform (see for example [35, 42, 48]), which are more or less equivalent to the one of Theorem 3. The first such formula has been derived by Johann Radon in 1917 a long time before the development of computed tomography (see [61]; the original paper is reprinted in the book [35, pp. 177–192]). Note that Cormack and Hounsfield, the inventors of CT, had originally been unaware of the work of Radon and therefore independently (of Radon and each other) derived appropriate inversion techniques.

Dual convolution

While the filtered backprojection algorithm can be seen as a numerical implementation of (12), it is more naturally developed using a convolution identity for the dual Radon transform we shall study next. For that purpose, we denote by $(f_1 * f_2)(x) := \int_{\mathbb{R}^d} f_1(x - y) f_2(y) \mathrm{d}y$ the convolution of two functions $f_1, f_2 \colon \mathbb{R}^d \to \mathbb{R}$. When applied to functions defined on $S^1 \times \mathbb{R}$, we make the convention that it only acts in the second component. Further, we define dual Radon transform

$$\left(\mathcal{R}^\sharp g\right)(x) = \int_{S^1} g(\theta, \langle\theta, x\rangle) \mathrm{d}\theta,$$

of some integrable function $g \colon S^1 \times \mathbb{R} \to \mathbb{R}$. One easily shows that one has in fact the following dual property

$$\int_{S^1} \int_{\mathbb{R}} (\mathcal{R} f)(\theta, s) g(\theta, s) \mathrm{d}s \mathrm{d}\theta = \int_{\mathbb{R}^2} f(x) \left(\mathcal{R}^\sharp g\right)(x) \mathrm{d}x.$$

Further, the following important properties hold, which serves as the basis of the filtered backprojection algorithm we derive in the next section.

Theorem 4 (Dual convolution)**.** *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be integrable and let $g\colon S^1 \times \mathbb{R} \to \mathbb{R}$ be C^1 with sufficient decay at infinity. Then*

(a) $\left(\mathcal{R}^\sharp g\right) * f = \mathcal{R}^\sharp (g * \mathcal{R} f)$

(b) For all $\xi \in \mathbb{R}^2$, we have $\left(\mathcal{F}\mathcal{R}^\sharp g\right)(\xi) = 2\,|\xi|^{-1}\,\mathcal{F}_2 g\,(\xi/|\xi|, |\xi|)$.

(Note that the function $\mathcal{R}^\sharp g$ is in general not integrable and therefore $\mathcal{F}\mathcal{R}^\sharp g$ has to be defined in sense of distributions.)

Proof. (a) The definition of the dual transform and Fubini's theorem show

$$\begin{aligned}
\left(\mathcal{R}^\sharp g\right) * f(x) &= \int_{\mathbb{R}^2} \int_{S^1} g\left(\langle \theta, x-y\rangle\right) \mathrm{d}\theta f(y) \mathrm{d}y \\
&= \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} g\left(\langle \theta, x\rangle - s\right) f(s\theta + t\theta^\perp) \mathrm{d}t \mathrm{d}s \mathrm{d}\theta \\
&= \int_{S^1} \int_{\mathbb{R}} g\left(\langle \theta, x\rangle - s\right) \mathcal{R} f(\theta, s) \mathrm{d}s \mathrm{d}\theta \\
&= \mathcal{R}^\sharp (g * \mathcal{R} f)(x).
\end{aligned}$$

(b) Suppose that $\varphi \in \mathcal{S}\left(\mathbb{R}^2\right)$ is any Schwartz function. According to the distributional definition of the Fourier transform and the Fourier slice theorem,

$$\begin{aligned}
\int_{\mathbb{R}^2} \left(\mathcal{R}^\sharp g\right)(x)\,(\mathcal{F}\varphi)(x) \mathrm{d}x &= \int_{S^1} \int_{\mathbb{R}} g(\theta, \sigma)\,(\mathcal{R}\mathcal{F}\varphi)(\theta, \sigma) \mathrm{d}\sigma \mathrm{d}\theta \\
&= \int_{S^1} \int_{\mathbb{R}} (\mathcal{F} g)(\theta, \sigma)\left(\mathcal{F}^{-1}\mathcal{R}\mathcal{F}\varphi\right)(\theta, \sigma) \mathrm{d}\sigma \mathrm{d}\theta \\
&= 2 \int_{S^1} \int_0 (\mathcal{F} g)(\theta, \sigma) \varphi(\sigma\theta) \mathrm{d}\sigma \mathrm{d}\theta \\
&= 2 \int_{\mathbb{R}^2} \frac{(\mathcal{F} g)(\xi/|\xi|, |\xi|)}{|\xi|} \varphi(\xi) \mathrm{d}\xi.
\end{aligned}$$

This shows that $\mathcal{F}\mathcal{R}^\sharp g$ is a regular distribution and represented by the function $\xi \mapsto 2\frac{(\mathcal{F} g)(\xi/|\xi|, |\xi|)}{|\xi|}$. □

Generalization to higher dimensions

The Radon transform can easily be generalized to higher dimensions, where it maps a function $f\colon \mathbb{R}^n \to \mathbb{R}$ to its integrals $(\mathcal{R} f)(\theta, s) = \int_{H(\theta,s)} f(x) \mathrm{d}S(x)$ over hyperplanes $H(\theta, s) = \{x \in \mathbb{R}^n : \langle \theta, x\rangle = s\}$, where θ is a normal vector of the hyperplane $H(\theta, s)$, s its oriented distance from the origin and $\mathrm{d}S$ denotes the $n-1$ dimensional surface measure. Most results for the two dimensional case generalize to higher dimensions as well (see, for example [35, 48, 50]).

For example, in three spatial dimensions, the analogon of the filtered back-projection inversion formula (12) reads

$$f(x) = -\frac{1}{8\pi^2} \int_{S^2} (\mathcal{R}f)''(\theta, \langle \theta, x \rangle) \mathrm{d}\theta \qquad \text{for } x \in \mathbb{R}^3, \tag{11}$$

where $(\mathcal{R}f)''$ denotes the second derivative of $\mathcal{R}f$ with respect to the second variable. One notices that the inversion formula for the Radon transform in three dimensions looks simpler than its analogon in two dimensions. Moreover, the inversion formula in three dimensions is local in the sense that recovering f at a single point $x \in \mathbb{R}^3$ using (11) only requires values of the Radon transform corresponding to planes which pass through an arbitrarily small neighbourhood of the reconstruction point x. Opposed to that, the 2D inversion formula (10) is non-local: Recovering f at a single point requires knowledge of the integrals of f over all lines in the plane. Note that such a discrepancy also holds in higher dimension: Inversion of the Radon transform is local in every odd dimension and non-local in every even dimension.

2.3 The filtered back-projection algorithm

The filtered backprojection (FBP) algorithm is still the most commonly used reconstruction algorithm for medical x-ray CT, see [58]. It may be seen as a computer implementation of the filtered backprojection inversion formula (10). However, due to the presence of the derivative, the inversion formula is sensitive to error in the data $\mathcal{R}f$. Such an instability is inherent in the Radon transform and one can show that inversion of the Radon transform is ill-posed of degree $1/2$ (see [48, Chap. II, Thm. 5.1]). For solving such an ill-posed problem one has to apply regularization techniques, which replace the exact solution by an approximate but stable one.

It is therefore reasonable derive the FBP algorithm from the already regularized formula (see Theorem 4)

$$\begin{aligned} (W_b * f)(x) &= \left(\mathcal{R}^\sharp (w_b * \mathcal{R}f) \right)(x) \\ &= \int_{S^1} \int_{\mathbb{R}} w_b(\theta, s) (\mathcal{R}f)(\theta, \langle \theta, x \rangle - s) \, \mathrm{d}s \, \mathrm{d}\theta, \end{aligned} \tag{12}$$

where $1/b > 0$ is a regularization parameter and $W_b \colon \mathbb{R}^2 \to \mathbb{R}$ and $w_b \colon S^1 \times \mathbb{R} \to \mathbb{R}$ satisfy the dual equation $W_b = \mathcal{R}^\sharp w_b$. In (12) the regularization effect comes from the convolution of the unknown f with a smooth radially symmetric mollifier $W_b \colon \mathbb{R}^2 \to \mathbb{R}$. If the family $\{W_b\}_{b>0}$ is such that $W_b * f \to f$ as $b \to \infty$, then $W_b * f$ is a smooth approximation of the unknown f that can be computed in stable way from the Radon data $\mathcal{R}f$.

The approximate inversion formula (12) is again of the filtered backprojection type. The inner operation is the convolution in the variable s with a smooth kernel and is referred to as the filtering step. The outer operation is refereed to as backprojection and integrates $w_b * \mathcal{R}f$ over all lines that pass through the reconstruction point x. The function w_b is the filtering kernel and requires solving the dual equation $W_b = \mathcal{R}^\sharp w_b$. A variety of filtering kernels can be designed using the following corollary of Theorem 4.

Corollary 5. *Suppose* $\Phi\colon [0,\infty) \to \mathbb{R}$ *is an integrable function satisfying* $0 \le \Phi \le 1$ *and* $\Phi(\sigma) = 0$ *for* $\sigma \ge 1$. *Further, let* W_b *denote the inverse Fourier transform of*

$$\hat{W}_b\colon \mathbb{R}^2 \to \mathbb{R}\colon \xi \mapsto \Phi\left(\frac{|\xi|}{b}\right). \tag{13}$$

Then (12) holds with

$$\mathcal{F}_2 w_b(\theta,\sigma) = \frac{|\sigma|}{2}\,\Phi\left(\frac{|\sigma|}{b}\right). \tag{14}$$

Proof. Defining W_b, w_b by their Fourier representations (13), (14), Item (b) in Theorem 4 shows $W_b = \mathcal{R}^\sharp w_b$ and therefore (12) follows from Theorem 4 (a). □

The standard FBP algorithm is a straightforward numerical implementation of (12). For that purpose, suppose that only discrete data

$$g_{j,k} := (\mathcal{R}f)(\theta_j, k\Delta s), \quad \text{for } (j,k) \in \{1,\dots,N\} \times \{-M,\dots,M\},$$

are given, where $\theta_j := (\cos\varphi_j, \sin\varphi_j)$ with $\varphi_j = 2(j-1)\pi/N$ and $\Delta s := 1/M$. The FBP algorithm uses the composite trapezoidal rule for discretizing the inner integral in (12) at the sampling points which yields

$$\Delta s \sum_{\ell=-M}^{M} w_b(k\Delta s - \ell\Delta s) g_{j,\ell} \simeq \int_{\mathbb{R}} w_b(s-t)\mathcal{R}f(\theta_j, s)\mathrm{d}s. \tag{15}$$

Notice that w_b is independent of $\theta \in S^1$, and we have dropped this first argument. The outer integration (backprojection operation) is also discretized with the composite trapezoidal rule evaluated at certain grid points $x \in D$. The required values of $(w_b * \mathcal{R}f)(\theta_j, \cdot)$ evaluated at $\langle \theta_j, x\rangle$ are computed with (15) and subsequent linear interpolation in the second argument.

The most critical step in the FBP algorithm is the discrete convolution in (15), defined by the discrete reconstruction filter

$$(w_b(k\Delta s))_{k=-M,\dots,M},$$

whose entries are samples of the filtering kernel w_b at the sampling points $k\Delta s$. Based on Corollary 5, one can derive most reconstruction filters used in CT. For

example, the strict low pass filter defined by $\Phi(\sigma) = 1$ for $\sigma \in [0,1]$ and $\Phi(\sigma) = 0$ otherwise, yields the filter coefficients

$$w_b(k\Delta s) = \frac{b^2}{2\pi^2} \begin{cases} 1/4 & \text{for } k = 0 \\ -1/(\pi^2 k^2) & \text{for } k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

This filter has been proposed in 1971 by Ramachandran and Lakshminarayanan [62] and is referred to as *Ram-Lak filter*. The choice $\Phi(\sigma) = \sin(\sigma\pi/2)/(\sigma\pi/2)$ for $\sigma \in [0,1]$ and $\Phi(\sigma) = 0$ otherwise has been proposed in 1974 by Shepp and Logan. The resulting *Shepp-Logan filter coefficients* are given by

$$w_b(k\Delta s) = \frac{b^2}{\pi^4} \frac{1}{1-4k^2}.$$

For more details on the FBP algorithm and filter design, see [10, 42, 48].

3 The spherical Radon transform

The classical Radon transform maps a function to its integrals over straight lines. As we have seen in the previous section, it serves as the basis of x-ray CT. In a number of different imaging technologies, there arises a need to reconstruct an unknown function from its integrals over spheres. This leads to the inversion of the so-called spherical Radon transform, which we study in this section.

For an integrable function $f\colon \mathbb{R}^n \to \mathbb{R}$, the spherical Radon transform $\mathcal{R}_{\text{sph}} f\colon \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ is defined by

$$(\mathcal{R}_{\text{sph}} f)(x,t) := \int_{\partial B(x,t)} f(y) \mathrm{d}s(y) \qquad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty).$$

Here $B(x,t) := \{y \in \mathbb{R}^n : |x-y| < t\}$ is the open n-dimensional ball of radius $t > 0$ centered at x with respect to the usual Euclidean norm, $\partial B(x,t)$ is its boundary, and $\mathrm{d}s$ denotes the standard surface measure. For $n = 2$, one also calls $\mathcal{R}_{\text{sph}}$ the circular Radon transform.

The spherical Radon transform arises, for example, in photoacoustic tomography (PAT) [7, 19, 44, 74], sound navigation and ranging (SONAR) [4, 60], synthetic aperture radar (SAR) [1, 63], ultrasound tomography [51, 52], and seismic imaging [6, 15]. In the following subsection, we show how the 2D and 3D spherical Radon transforms arise in the quite recently developed PAT.

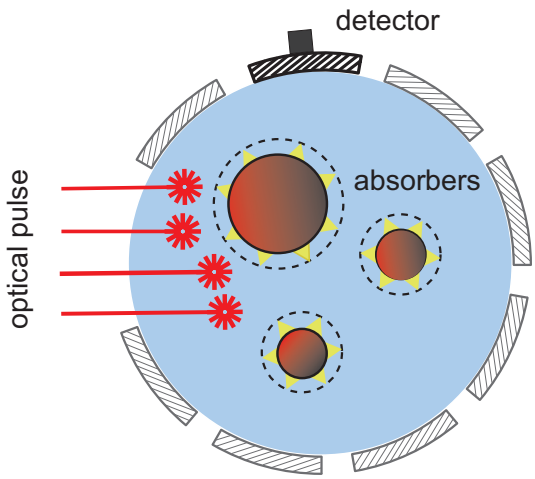


Figure 3: Illustration of PAT. Electromagnetic pulses are delivered into the tissue. Ultrasound detectors measure the generated acoustic waves on the boundary.

3.1 Photoacoustic tomography

PAT is based on the so-called photoacoustic effect. When short pulses of non-ionising electromagnetic energy are delivered into a biological (semi-transparent) tissue, then parts of the electromagnetic energy become absorbed. The absorbed energy leads to a nonuniform thermoelastic expansion (depending on the tissue structure), which in turn generates an ultrasonic wave. These waves are detected by a measurement device on the boundary of the tissue (see Figure 3). The mathematical task in PAT is to reconstruct the spatially varying absorption coefficient using these measurements.

While x-ray CT has a rather low contrast in soft tissues, the electromagnetic absorption coefficient at some lower frequencies shows significantly higher variation. PAT therefore provides good imaging contrast in soft tissues making it a very promising technique for detecting various types of early cancer, such as breast cancer or skin melanoma. In 1998, the first clinical prototype of a PAT scanner for breast screeing has been developed by Kruger [43]. Various practical aspects of PAT are discussed in [74].

The reconstruction of the absorption coefficient in an object under investigation from the measured acoustic waves on the boundary of the object requires a mathematical model for the relationship between the absorption coefficient and the boundary acoustic waves. Below we briefly review such a model, following the approach presented in [33, 66].

Mathematical modeling

Suppose the object to be investigated is supported in a domain $\Omega \subset \mathbb{R}^3$ that is illuminated with a short pule of electromagnetic energy near the visible range. We denote by $I(x,t) = J(x)j(t)$ the intensity of the electromagnetic energy at location $x \in \mathbb{R}^3$ and time $t \in \mathbb{R}$ with $J(x)$ being the spatial and $j(t)$ the temporal intensity distribution. The rate of absorbed electromagnetic energy is described by the absorbed electromagnetic power $r(x,t) = \mu_{\text{abs}}(x)I(x,t)$, where $\mu_{\text{abs}}(x)$ is the spatially varying absorption coefficient. The rate of absorbed energy causes a temperature change which is in turn related to an increase of acoustic pressure $p(x,t)$. Since the pulse duration is very short, the heat transfer by conduction can be neglected [69].

Employing the linearised equations of fluid dynamics (for details, see [27] or [66, Section 1.5]) one shows

$$\frac{1}{v_s(x)^2}\frac{\partial^2 p(x,t)}{\partial t^2} - \Delta p(x,t) = f(x)\frac{dj}{dt}(t)\,, \qquad \text{for } (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{16}$$

with $f(x) = I(x)\beta(x)\mu_{\text{abs}}(x)/C_p(x)$. Here $C_p(x)$ is the specific heat capacity, $\beta(x)$ is the thermal expansion coefficient at constant pressure and $v_s(x)$ is the speed of sound. The wave equation (16) is augmented with the initial conditions $p(x,t) = 0$ for $t < 0$, reflecting the fact that there is no acoustic pressure before the illumination starts at $t = 0$.

In the following we assume that the sound speed $v_s = v_s(x)$ is constant, and after rescaling we can assume that v_s equals one, and that $j(t)$ is approximates the one-dimensional δ-distribution. Then, by Duhamel's principle [14, p. 81], the solution of (16) coincides, for $t > 0$, with the solution of the initial value problem

$$\begin{cases} \left(\partial_t^2 - \Delta\right) p(x,t) = 0 & \text{for } (x,t) \in \mathbb{R}^3 \times (0,\infty) \\ p(x,0) = f(x) & \text{for } x \in \mathbb{R}^3 \\ \dfrac{\partial p}{\partial t}(x,0) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases} \tag{17}$$

The aim of PAT is to reconstruct the function $f(x)$, proportional to the absorption coefficient, from measurements of the solution of (17) taken outside of the support of f. The particular mathematical problem to solve also depends on the way how the acoustic signals are measured. Different measurement setups lead to different mathematical problems. Below we shortly review the concepts of point-like and linear integrating measurement setups, which yield to the inversion of the spherical Radon transform in three and two dimensions, respectively.

Point measurement: 3D spherical Radon transform

In the standard measurement procedure used in PAT, small piezoelectric detectors are placed on the object's boundary and they record arriving acoustic waves there (see Figure 4 left). These detectors can be seen as an approximation to idealised point detectors that record the solution of (17) pointwise on the boundary $\partial\Omega$.

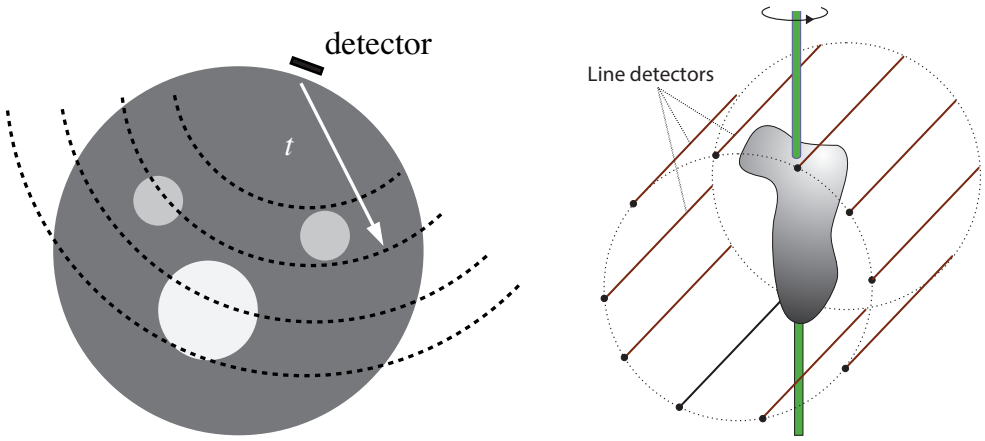


Figure 4: LEFT: The data measured by small piezoelectric detectors can provide integrals of the unknown function over spheres. RIGHT: The array of the line detectors measures the acoustic waves during the rotation around the object.

The well known explicit expression for the solution of the initial value problem (17) in terms of the three-dimensional spherical Radon transform (see, for example, [14, page 72]) reads

$$p(x,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}(\mathcal{R}_{\text{sph}}f)(x,t)\right] \qquad \text{for } (x,t) \in \mathbb{R}^3 \times (0,\infty).$$

Integrating this expression with respect to t yields

$$(\mathcal{R}_{\text{sph}}f)(x,t) = 4\pi t \int_0^t p(x,s)\mathrm{d}s.$$

Thus, the reconstruction of the initial pressure distribution from measurements of point detectors on the object boundary yield to the problem of inverting the spherical Radon with centers restricted to $\partial\Omega$.

Line measurement: 2D spherical Radon transform

Since in practice every acoustic detector has a finite size, the algorithms that are based on the assumption of point-like measurements produce blurred reconstruc-

tion (see [34, 64, 72]). In order to partially compensate for this effect, in [32] it is suggested to use large planar detectors that measure integrals of the acoustic pressure over planes. As a further development in [8, 57] so-called line detectors have bee proposed that use integrals of the acoustic pressure over lines and which can be efficiently realized in practice.

The measurement setup with line detectors is as follows. The devices are put into an array of detectors that are parallel to each other, and this array is rotated around a single axis (Figure 4 right). Let p be the solution of (17) and assume that the line detectors are parallel to the direction $e_1 := (1,0,0)$. Also, let us write $x = (x_1, x')$ with $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^2$, and denote by

$$\bar{p}(x',t) = \int_{\mathbb{R}} p(x_1,x',t)\mathrm{d}x_1 \quad \text{for } (x',t) \in \mathbb{R}^2 \times (0,\infty)$$

the pressure values integrated in direction e_1. It is not hard to show that the integrated pressure $\bar{p}$ satisfies the following two dimensional initial value problem (see, for example, [7])

$$\begin{cases} \left(\partial_t^2 - \Delta\right)\bar{p}(x',t) = 0 & \text{for } (x',t) \in \mathbb{R}^2 \times (0,\infty) \\ \bar{p}(x',0) = \bar{f}(x') & \text{for } x' \in \mathbb{R}^2 \\ \dfrac{\partial \bar{p}}{\partial t}(x',0) = 0 & \text{for } x' \in \mathbb{R}^2. \end{cases} \tag{18}$$

Here $\bar{f}(x') := \int_{\mathbb{R}} f(x_1,x')\mathrm{d}x_1$ is the linear projection of f in direction e_1. Data of line detectors values $p(x',t)$ for certain measurement positions x' outside of the support of f'. Note that having obtained the linear projections $\bar{f}$ from different directions, the reconstruction of f can be obtained from the inversion of the classical (linear) Radon transform studied in Section 2.

Similar to its three dimensional counterpart, the two dimensional reconstruction problem based on (18) can be recast as the problem of inverting the spherical Radon transform: Note that the solution of the 2D wave equation is given by (see, for example, [40, Equation (1.24a)])

$$\bar{p}(x',t) = \frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^t \frac{\left(\mathcal{R}_{\mathrm{sph}}\bar{f}\right)(x',r)}{\sqrt{t^2-r^2}}\,\mathrm{d}r \quad \text{for } (x',t) \in \mathbb{R}^2 \times (0,\infty).$$

Application of standard tools for solving Abel type integral equations (see, for example, [23, 48]) yields the following expression for $\mathcal{R}_{\mathrm{sph}}\bar{f}$ in terms of the data values:

$$\left(\mathcal{R}_{\mathrm{sph}}\bar{f}\right)(x',r) = 4r\int_0^r \frac{\bar{p}(x',t)}{\sqrt{r^2-t^2}}\mathrm{d}t.$$

Consequently, PAT with integrating line detectors yields to the problem of reconstructing $\bar{f}$ from its circular Radon transform, which is the 2D analogon of the 3D reconstruction problem in PAT using point-like measurements.

3.2 Inversion formulas

Exact inversion formulas for the spherical Radon transform are currently known for boundaries of special domains, including spheres, cylinders and hyperplanes. Recently explicit inversion formulas for elliptic domains started to appear in the literature [2, 28, 30, 49, 54]. In [31], we showed that the formula [28] for elliptic domains in 2D is also exact for parabolic domains.

Exemplarily we present inversion formulas from [17] for the circular Radon transform with centers of integration restricted to a circle, and the inversion formulas from [28] for boundaries of ellipses, which are also exact for circles.

Recall that $B(x,t) \subset \mathbb{R}^2$ denotes the open ball of radius $t > 0$ centered at x. For a general domain $\Omega \subset \mathbb{R}^2$, we denote by $C_c^\infty(\Omega)$ the set of all smooth functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ that are compactly supported in Ω.

Theorem 6 (Inversion formulas of [17])**.** *Let $D_R := B(0,R) \subset \mathbb{R}^2$ denote the disc of radius R centered at the origin, suppose that $f \in C_c^\infty(D_R)$ and extend $(\mathcal{R}_{\text{sph}} f)(x,t)$ as an even function in the second variable t.*
Then, for all $x_0 \in D_R$, the function f can be recovered from $\mathcal{R}_{\text{sph}} f$ with the help of the following formulas:

$$f(x_0) = \frac{1}{4\pi^2 R} \int_{\partial D_R} \int_{-2R}^{2R} \frac{(t\partial_t t^{-1} \mathcal{R}_{\text{sph}} f)(x,t)}{|x_0 - x| - t} \, \mathrm{d}t \mathrm{d}s(x),$$
$$f(x_0) = \frac{1}{4\pi^2 R} \int_{\partial D_R} |x_0 - x| \int_{-2R}^{2R} \frac{(\partial_t t^{-1} \mathcal{R}_{\text{sph}} f)(x,t)}{|x_0 - x| - t} \, \mathrm{d}t \mathrm{d}s(x),$$

where the inner integrals are taken in the principal value sense.

Note that many researchers believed that exact reconstruction formulas in 2D exist only for circles and lines. However recently, it was shown in [49] that the so-called universal back-projection formula from [73] is theoretically exact for ellipsoids in $\mathbb{R}^3$. In [28] such formulas have been derived for ellipses in $\mathbb{R}^2$. The formulas of [49, 28] in fact can be used for arbitrary bounded convex domains. However, in this case the formulas do not recover the underlying function exactly and give an error. In both papers [49, 28], the corresponding error term has been explicitly derived. The results of [28] have been generalized to arbitrary dimension in [30]. In $\mathbb{R}^3$, the corresponding formulas coincide with the formulas from [49]. Note that for the special case of spherical domains the formulas of [30] also coincide with the formulas of [45]. Very recently one of the formulas of [17, 18] has been generalized to elliptical domains in [65, 29].

The inversion formulas from [28] read as follows.

Theorem 7 (Inversion formulas of [28])**.** *Suppose $\Omega \subset \mathbb{R}^2$ is a circular or ellip-*

tical domain and let $f \in C_c^\infty(\Omega)$. Then, for all $x_0 \in \Omega$, the following holds:

$$f(x_0) = \frac{1}{2\pi^2} \nabla_{x_0} \cdot \int_{\partial\Omega} \nu_x \int_0^\infty \frac{\mathcal{R}_{\text{sph}} f(x,t)}{t^2 - |x_0 - x|^2} \, \mathrm{d}t \mathrm{d}s(x),$$

$$f(x_0) = \frac{1}{2\pi^2} \int_{\partial\Omega} \langle \nu_x, x_0 - x \rangle \int_0^\infty \frac{(\partial_t t^{-1} \mathcal{R}_{\text{sph}}) f(x,t)}{t^2 - |x_0 - x|^2} \, \mathrm{d}t \mathrm{d}s(x). \tag{19}$$

Here ν_x denotes the outwards pointing unit normal to $\partial\Omega$ and the inner integrals are understood in the principal value sense.

In [31], we showed that the second formula in the above theorem is also exact for the case where Ω is a parabolic domain.

If one has the pure wave data $\bar{p}(x,t)$, i.e. the solution of the initial value problem (18) is given on the boundary of an elliptic or parabolic domain Ω, then the corresponding initial pressure distribution can be recovered by means of the following formula (see [7, 28, 31]):

$$\bar{f}(x_0) = \frac{1}{\pi} \int_{\partial\Omega} \langle \nu_x, x_0 - x \rangle \int_{|x_0 - x|}^\infty \frac{\left(\partial_t t^{-1} \bar{p}\right)(x,t)}{\sqrt{t^2 - |x_0 - x|^2}} \, \mathrm{d}t \mathrm{d}s(x), \tag{20}$$

for any reconstruction point $x_0 \in \Omega$.

3.3 Numerical results

The formulas (19), (20) can be implemented as outlined in [7, 17]. For illustration, we present numerical results for the recovery from the wave data (the solution (18)) in two spatial dimensions. We consider a function $\bar{f}\colon \mathbb{R}^2 \to \mathbb{R}$ given by the phantom shown in Figure 5. The same phantom has been used for testing the numerical performance of the reconstruction formulas in [7, 28, 31]. The support of the corresponding function $\bar{f}$ is included in the parabolic domain

$$P = \left\{ (a,b) \in \mathbb{R}^2 \mid b > 0.6\, a^2 - 1 \right\}.$$

For the numerical realization of formula (20), one first has to replace the integral over ∂P by the integral over a curve with finite length. We take the following integration curves:

$$\Gamma_i = \left\{ (a,b) \in \mathbb{R}^2 \mid b = 0.6\, a^2 - 1,\ a \in [-a_i, a_i] \right\} \qquad \text{for } i = 1,2,3,$$

with $a_1 = 2$, $a_2 = 4$, $a_3 = 6$, respectively. The simulated wave data $\bar{p}$ on the curve Γ_1 is presented in Figure 5.

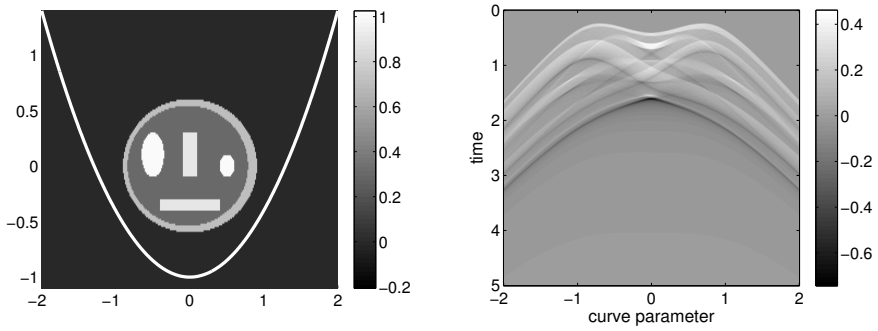


Figure 5: LEFT: The phantom in the parabolic domain P that is used for the numerical results. RIGHT: The simulated wave data $\bar{p}$ on the recording curve Γ_1. The variable a is considered as the curve parameter.

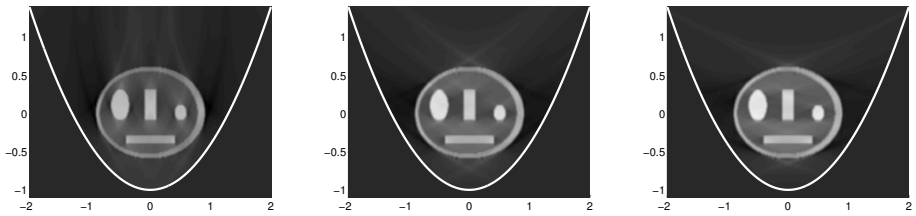


Figure 6: The numerical reconstructions $\bar{f}_i$, $i = 1,2,3$ (from left to right) on the reconstruction subdomain $\bar{P}$. The gray scale is as for the phantom of Figure 5.

Let us present the reconstructions $\bar{f}_i(x)$ that are obtained by the numerical realization of formula (20) where the integration curve ∂P is replaced by Γ_i. These reconstructions on the reconstruction subdomain (the set where the inversion formula is evaluated)

$$\bar{P} = \left\{(a,b) \in \mathbb{R}^2 \mid 0.6a^2 - 1 < b < 0.6 \cdot 2^2 - 1,\ a \in (-2,2)\right\}$$

at the points $\left\{0.015(i,j) : (i,j)\right\} \in \mathbb{Z}^2\big\} \cap \bar{P}$ are shown in Figure 6. The time step size for the inner integral in (20) is taken 0.01. The integration curves are discretized such that the distance between two consecutive points is in the interval $[0.0099, 0.0101]$. The numbers of the discretization points on the considered integration curves are the following: 659, 2166, 4617.

The reconstruction errors for the finite parabolas Γ_i decrease as the length of Γ_i increases. It should be noted that the reconstruction problem in the case of the open curves Γ_i corresponds to the so-called limited view problem [46, 55, 56, 75]. For each reconstruction point inside the reconstruction subdomain $\bar{P}$ there is a

considerable set of directions for which the boundary wave data is missing, which is known to create reconstruction artefacts. We refer to [31] for the comparison with the reconstructions on closed parabolas and ellipses.

4 Concluding remarks

In this note we have given a brief introduction to CT and presented mathematical results serving as basis of FBP reconstruction algorithms. We thereby focused on two prime examples, namely classic x-ray CT and the more recent PAT. These applications require inversion of the classical Radon transform and the spherical Radon transform, respectively. The most common algorithm for these applications is the filtered backprojection (FBP) algorithm, which implements exact inversion formulas we presented above.

Note that we mainly focused on the development of FBP algorithms for inverting Radon type transforms. In some tomographic applications iterative reconstruction algorithms are more common. For example, in single photo emission tomography (SPECT), the statistical noise is an important issue, and iterative reconstruction algorithms based on a maximum likelihood minimization are favoured. A prominent iterative procedure for maximum likelihood minimization is the EM algorithm (expectation maximisation algorithm) of Dempster, Laird and Rubin [11], which has been introduced to computed tomography in [68, 71]. Note that also the first reconstruction algorithms in x-ray CT have been of iterative nature (see [37, 22]). The used algorithm became popular known under the name ART (Algebraic Reconstruction Technique) and was later (see [26]) identified as Kaczmarz's iterative procedure [41] for the solution of system of linear equations. See, for example, [36, 48, 50] for more details on the use of iterative reconstruction algorithm in CT.

Finally, note that some tomographic application are better modelled as parameter identification problems for partial differential equations. This often yields to nonlinear inverse problems. See [13, 38, 39, 47, 66, 70] for general solution methods approaching such type of problems.

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Buchbesprechungen

C. Alsina, R. B. Nelsen: Bezaubernde Beweise (S. WAGNER) 54
L. Barreira, Y. Pesin: Introduction to Smooth Ergodic Theory (C. PÖTZSCHE) 54
E. Behrends: Fünf Minuten Mathematik. 100 Beiträge der Mathematik-Kolumne der Zeitung *Die Welt* (G. HARING) 55
A. Bonato, R. J. Nowakowski: The Game of Cops and Robbers on Graphs (W. IMRICH) 56
C. Cottin, S. Döhler: Risikoanalyse (M. PREDOTA) 57
H. Dym: Linear Algebra in Action (G. TESCHL) 57
O. Forster, T. Szymczak: Übungsbuch zur Analysis 2. Aufgaben und Lösungen (C. ELSHOTZ) 58
D. W. Hoffmann: Grenzen der Mathematik (C. ELSHOLTZ) 58
J.-M. De Koninck, F. Luca: Analytic Number Theory (T. STOLL) 59
B. Landman, M. B. Nathanson et al. (eds): Combinatorial Number Theory. Proceedings of the "Integers Conference 2011", Carrollton, Georgia, Oct. 26–29, 2011 (C. ELSHOLTZ) 59
L. Lovász: Large Networks and Graph Limits (S. WAGNER) 60
B. Luderer: Mathe, Märkte und Millionen. Plaudereien über Finanzmathematik zum Mitdenken und Mitrechnen (W. JANKO) 60
K. Nipp, D. Stoffer: Invariant Manifolds in Discrete and Continuous Dynamical Systems (C. PÖTZSCHE) 61
T. Tao: Topics in Random Matrix Theory (A. R. KRÄUTER) 62
M. Willem: Functional Analysis. Fundamentals and Applications (G. TESCHL) 63

C. Alsina, R. B. Nelsen: Bezaubernde Beweise. Eine Reise durch die Eleganz der Mathematik. Aus dem Englischen von T. Filk. Springer Spektrum, Springer Berlin, Heidelberg, 2013, xiv+326 S. ISBN 978-3-642-34792-4 P/b € 23,32.

Bezaubernde Beweise hält, was der Buchtitel verspricht: Das Buch ist voll mit eleganten und instruktiven Beweisen aus verschiedensten Gebieten der Mathematik, darunter viele bekannte Klassiker wie die Divergenz der harmonischen Reihe, die Irrationalität von $\sqrt{2}$, e und π, der kleine Satz von Fermat oder die Eulersche Polyederformel.

Die weitaus meisten Sätze, die bewiesen werden, sind elementarer Natur und benötigen nur Schulkenntnisse der Mathematik. Besonders stark ist dabei die Geometrie in all ihren Ausprägungen (Euklidische Geometrie, Zerlegungsgeometrie, spezielle Kurven, dreidimensionale Geometrie, etc.) vertreten. Jedes Kapitel ist zudem mit einer Liste von Übungsaufgaben versehen, anhand derer man die eigenen Fertigkeiten testen kann.

Das Buch eignet sich ausgezeichnet, um vor allem junge Leute für die Eleganz und Schönheit mathematischer Beweise zu begeistern. Durch den relativ elementaren Inhalt kann man es einem sehr breiten Publikum empfehlen, insbesondere auch Nichtmathematikern, die in die Welt der Beweise eintauchen wollen.

S. Wagner (Stellenbosch)

L. Barreira, Y. Pesin: Introduction to Smooth Ergodic Theory. (Graduate Studies in Mathematics, Vol. 148.) American Mathematical Society, Providence, Rhode Island, 2013, ix+276 S. ISBN 978-0-8218-9853-6 H/b $ 65, € 47,–.

The book at hand provides an introduction to smooth ergodic and nonuniform hyperbolicity theory. It represents a revised and largely expanded version (100+ pages) of the authors' *Lyapunov Exponents and Smooth Ergodic Theory* (Univ. Lect. Ser. 23, AMS, Providence RI, 2002) and in turn their 1999 survey on the topic. In particular, it contains a systematic account of Pesin's original four papers from the 1970s establishing the deep theory of nonuniformly hyperbolic smooth dynamical systems including detailed proofs and well-chosen examples.

In the first part, aptly titled "Core of the Theory", the authors begin with an informative and motivating chapter containing various examples of hyperbolic systems including Anosov diffeomorphisms and flows, Katok's map on the 2-torus and systems with nonzero Lyapunov exponents on surfaces. An elegant introduction to the finite-dimensional theory of Lyapunov exponents, based on three axioms, is presented in Chapter 2; it includes a discussion of the central regularity notions and also describes the corresponding theory in discrete time. Afterwards applications to the stability theory of nonautonomous ODEs are given. Here, nonetheless some information on Bohl exponents yielding the arguably more appropriate uniform stability notions would have been desirable. Elements of the "Nonuniform

Hyperbolicity Theory" (systems with nonzero exponents, regular sets, partial hyperbolicity and the Hölder continuity of invariant distributions) are discussed, as well as "Cocycles over Dynamical Systems". As a matter of course, one highlight is a proof and the treatment of Oseledets' celebrated Multiplicative Ergodic Theorem. The following chapters deal with local invariant manifolds, which are constructed using a Lyapunov-Perron approach and a quantitative version of the implicit mapping theorem, foliations, and their absolute continuity. Further topics include ergodic properties of smooth hyperbolic measures and geodesic flows on surfaces, as well as their ergodic properties.

The second part deals with "Selected Advanced Topics". Cone techniques are suggested to construct Lyapunov functions guaranteeing nonzero exponents. Also partially hyperbolic diffeomorphisms are addressed, and the authors return to foliations lacking absolute continuity. Hyperbolic diffeomorphisms with countably many ergodic components and the Shub-Wilkinson map serve as further examples for the occurrence of nonzero exponents. After a chapter on Anosov rigidity, Pugh's example of C^1-pathological behavior completes the presentation.

Although not the only monograph in this area, without question the widely self-contained book can be warmly recommended for an advanced course on Dynamical Systems or Ergodic Theory; it contains over 80 exercises. The advanced topics might also be appropriate for a subsequent seminar.

Ch. Pötzsche (Klagenfurt)

E. Behrends: Fünf Minuten Mathematik. 100 Beiträge der Mathematik-Kolumne der Zeitung *Die Welt*. Mit Geleitwort von N. Lossau, Springer Spektrum, Springer Fachmedien, Wiesbaden, 2013, xvii+262 S. ISBN 978-3-658-00998-4 P/b € 25,65.

Am 12. Mai 2003 startete der Autor in der überregionalen Zeitung „*Die Welt*" eine regelmäßige Kolumne zur Mathematik. In den darauffolgenden zwei Jahren entstanden so 100 Beiträge zu sehr unterschiedlichen mathematischen Themen, die ein breites Spektrum abdecken. Die Zielsetzung war es, über mathematische Fragestellungen in einer ansprechenden Form, verpackt in spannenden und motivierenden Geschichten aus dem wirklichen Leben, zu informieren – ohne große fachliche Kenntnisse beim Leser voraussetzen zu müssen. Auf diese Weise sollte, in knapper und verständlicher Form, gezeigt werden, wie Mathematik unser Leben durchdringt. Dem Autor war es dabei wichtig, zu zeigen, dass Mathematik nützlich und faszinierend ist, und dass man ohne Mathematik die Welt nicht wirklich verstehen kann. Durch die Übernahme dieser Kolumnen in Buchform sollte eine weitere Leserschaft erschlossen werden; das vorliegende Werk ist bereits die 3. Auflage. Für die Buchform wurden die Beiträge überarbeitet und ergänzt, z.B. durch Bilder, Illustrationen, erläuternde Texte und Filme (auf *YouTube*), was letztlich den Umfang mehr als verdoppelte. Dieser sehr ansprechende Ansatz hat auch

international ein breites Echo gefunden; das Buch wurde bereits in mehrere Sprachen übersetzt. Der Grad der Zufriedenheit des Lesers hängt sicher davon ab, was dieser sich erwartet und wie weit er auch bereit und offen ist, sich in die faszinierende Welt mathematischer Aspekte derselben führen zu lassen. Das Werk stellt ohne Zweifel einen wertvollen Beitrag dar, die Mathematik einem breiteren Publikum auf erbauliche Weise näherzubringen.

G. Haring (Wien)

A. Bonato, R. J. Nowakowski: The Game of Cops and Robbers on Graphs. (Student Mathematical Library, Vol. 61.) American Mathematical Society, Providence, Rhode Island, 2011, xix+276 S. ISBN 978-0-8218-5347-4 P/b $ 45,–.

This is a delightful book about a simple game. The game is played by two players. One player controls the cops, who are trying to catch the robber, and the other controls the robber. Cops and robbers are restricted to the vertices of a finite or infinite graph and move in each round to neighboring vertices. The smallest number of cops needed to catch the robber is the *cop number*.

The prerequisites of the book are basic graph theory, some mathematical maturity and background in sets, probability and algorithms. Nonetheless, it leads to or touches areas such as complexity, probabilistic arguments, homomorphisms, products of graphs, and the structure of finite and infinite graphs.

The first chapter introduces the game, supplies notation, basic results, examples, and provides the motivation to what comes later. It also gives an asymptotic upper bound on the cop number. The second chapter characterizes k-cop win graphs, which requires retractions, vertex orderings, relations and graph products. The third chapter discusses Meyniel's Conjecture that $O(\sqrt{n})$ cops suffice to catch a robber on a graph with n vertices. Here randomized methods are introduced. This provides the basis for Cops and Robbers on random graphs, which is the topic of chapter six. Chapter four considers the cop number in graph products and graph classes. For example, it is shown that the cop number in planar graphs is at most three. Graphs of higher genus are also discussed. Chapter five treats algorithms and complexity issues, and chapter seven cops and robbers in infinite graphs. It is known that infinite graphs exhibit unusual properties not seen in finite graphs; this is also true for the cop number. For example, paradoxically large families of infinite vertex-transitive graphs are constructed via weak strong products of graphs.

In chapter eight the rules of the game are changed, but the core of the game remains that a set of *good guys* tries to capture, stop, or contain a set of *bad guys*. Several of these games are then treated in chapter nine.

The book contains a large selection of worked out examples and exercises. It could be the basis of a good second course in graph theory.

W. Imrich (Leoben)

C. Cottin, S. Döhler: Risikoanalyse. Modellierung, Beurteilung und Management von Risiken mit Praxisbeispielen. (Studienbücher Wirtschaftsmathematik) Springer Spektrum, Springer Fachmedien, Wiesbaden, 2013, xviii+456 S. ISBN 978-3-658-00830-7 P/b € 37,34.

Das vorliegende Buch gibt einen sehr praxisorientierten Überblick zu verschiedenen Themen aus der Risikoanalyse und dem Risikomanagement und basiert auf Lehrveranstaltungen der Autoren. Dabei werden auch viele konkrete Beispiele vorgeführt, was die Verknüpfung zwischen Theorie und Anwendung schließt. Das Buch unterstützt dabei sowohl die Statistiksoftware *R* als auch das Tabellenkalkulationsprogramm *Microsoft Excel*.

Einführend beginnt das Buch mit einem geschichtlichen Überblick der Risikoanalyse sowie gesetzliche Rahmenbedingungen (u.a. Basel III, Solvency II) und führt etliche Verteilungen ein, die für die Modellierung von Risiken verwendet werden können. In weiterer Folge wird auf einfache Risikokennzahlen, Risikoentlastungsstrategien, Auswahl und Entwicklung von Modellen und Simulationsmethoden eingegangen. Die vorliegende 2. Auflage ist vor allem um Aspekte der Extremwerttheorie sowie der Zerlegung ausgewählter strukturierter Finanzprodukte (Zertifikate) ergänzt. Jedes Kapitel des Buchs schließt mit Aufgaben und einem Selbsttest. Das Buch kann als Lehrbuch für einführende Vorlesungen sowie auch für das Selbststudium (bei geringem bis keinem Vorwissen) im Risikomanagement und der Risikoanalyse empfohlen werden, mathematische Grundkenntnisse aus den ersten beiden Semestern sollten vorhanden sein.

M. Predota (Wien)

H. Dym: Linear Algebra in Action. (Graduate Studies in Mathematics, Vol. 78.) American Mathematical Society, Providence, Rhode Island, 2013, xix+585 S. ISBN 978-1-4704-0908-1 H/b $ 91, € 65,60.

Around here, the two main courses to master in your first semester are analysis and linear algebra. Usually, except for the real numbers, these two courses have nothing in common. The present textbook does not follow this philosophy and according to the reviews of the first addition this approach was well received.

The second edition has been extensively revised plus there is new material on linear programming, extreme points for polyhedra and a Nevanlinna-Pick interpolation problem, the mathematics behind Google, Drazin inverses, band inverses and applications of the singular value decomposition together with several new exercises.

In summary, a wonderful book has become even better.

G. Teschl (Wien)

O. Forster, T. Szymczak: Übungsbuch zur Analysis 2. Aufgaben und Lösungen. Springer Spektrum (Grundkurs Mathematik), Springer Fachmedien, Wiesbaden, 2013, viii+192 S. ISBN 978-3-658-00335-7 P/b € 16,40.

Dies Buch enthält Aufgaben und Lösungen zu dem Buch Analysis 2 von Otto Forster. Die Analysisreihe von Forster ist seit Jahrzehnten erfolgreich und hat Spitzenauflagen erzielt. Dies Übungsbuch enthält auf ca. 30 Seiten Aufgaben, und auf ca. 120 Seiten ausführliche Lösungen von fast allen Aufgaben. Die behandelten Themen sind insbesondere Differentialrechnung in mehreren Variablen und Differentialgleichungen. Die Aufgaben sind klassisch, so wie sie jeder Studierende in dieser oder ähnlichen Form einmal bearbeitet haben sollte. In dieser aktuellen 8. Auflage sind einige Aufgaben und Lösungen bearbeitet oder ergänzt worden.

C. Elsholtz (Graz)

D. W. Hoffmann: Grenzen der Mathematik. Eine Reise durch die Kerngebiete der mathematischen Logik. (Springer Spektrum) Berlin 2013, ix+438 S. ISBN 978-3-642-34719-1 € 30,83.

Dies Buch ist eine sehr schöne Einführung in die mathematische Logik. Das Thema ist vermutlich als „trocken“ gefürchtet, hier aber wird es durch den Schreibstil und die Zusatzinformationen geradezu spannend!

Zu den behandelten Themen gehören: Geschichte der mathematischen Logik, formale Systeme, axiomatische Zahlentheorie und Mengenlehre, Beweistheorie, Gödelsche Unvollständigkeitssätze, Berechenbarkeitstheorie, algorithmische Informationstheorie und Modelltheorie.

Dem Autor gelingt es, zu dem klassischen Material im Haupttext eine Fülle von zusätzlichen Materialien, wie z.B. Abbildungen und Fotos, Ausschnitte der Originalarbeiten, historische Bemerkungen und Übungsaufgaben, einzuarbeiten.

Das Buch will im Zweifelsfall eher die Ideen erklären, als mit der Fülle von Details erschlagen, für die dann auf weiterführende Literatur verwiesen wird. Zu den Übungsaufgaben gibt es auf einer Webseite Lösungen, die man sich einzeln (!) herunterladen kann. Dies ist eine interessante Idee, um das gedankenlose, massenhafte Kopieren der Lösungen zu verhindern.

Alles in allem ein sehr schönes Buch!

C. Elsholtz (Graz)

J.-M. De Koninck, F. Luca: Analytic Number Theory. Exploring the Anatomy of Integers. (Graduate Studies in Mathematics, Vol. 134.) American Mathematical Society, Providence, Rhode Island, 2012, xviii+414 S. ISBN 978-0-8218-7577-3 H/b $ 75,–.

There exists a number of excellent books on analytic number theory; for instance, Davenport's "Multiplicative Number Theory", G. Tenenbaum's "Introduction à la théorie analytique et probabiliste des nombres", Iwaniec/Kowalski's "Analytic Number Theory" and Montgomery/Vaughan's "Multiplicative Number Theory I: Classical Theory". The book under review is a down-to-earth introduction to the various questions and problems arising in analytic number theory whereby the main focus is on an elementary and easily accessible presentation of the material. A great asset to the text are the 263 exercises: The reader can find the solutions of all the even-numbered problems in the book. Students will greatly benefit from working through the exercises of this book.

The book starts with recalling a few preliminary techniques from real analysis (Chapter 1). The authors then develop the elementary prime number theory (Chebyshev estimates) in Chapter 2; Chapters 3 to 5 are devoted to the Newman proof for the prime number theorem (without error term), all tools from complex analysis are recalled in the Appendix. Chapter 6 and 7 treat the global/local behavior of arithmetic functions (Wintner's theorem, Turán-Kubilius inequality). Chapter 8 gives a primer on the distribution of the values of Euler's function, Chapter 9 deals with smooth numbers. Chapter 10 is on the Hardy-Ramanujan inequality and on Landau's theorem. In Chapter 11 the authors explain the *abc* Conjecture, and Chapter 12 introduces sieve methods (Brun sieve, Selberg sieve, Large sieve) with applications (Brun-Titchmarsh theorem etc.). The reader is sometimes referred to the literature for the proofs of deep theorems (Friedlander-Iwaniec theorem etc.). Chapters 13 to 15 treat the problem of counting prime numbers in arithmetic progressions, and various applications are given. In the final chapter (Chapter 16) the authors study the behavior of the index of composition of an integer, a notion introduced by De Koninck and Doyon in 2003.

The overall presentation is remarkably fresh and therefore best suited to motivated undergraduate or graduate students with interests in number theory.

T. Stoll (Nancy)

B. Landman, M. B. Nathanson et al. (eds): Combinatorial Number Theory. Proceedings of the "Integers Conference 2011", Carrollton, Georgia, Oct. 26–29, 2011. (Proceedings in Mathematics 11.) De Gruyter, Berlin/Boston, 2013, ix+157 S. ISBN 978-3-11-028048-7 H/b € 119,95.

Dies sind die Proceedings der "Integers Conference 2011". Der Band enthält zehn Beiträge, die ein breites Spektrum abdecken, von additiven und multiplikativen Problemen in der Zahlentheorie über Färbungsprobleme bis hin zu zwei Arti-

keln über kombinatorische Spiele. Die Artikel richten sich an Forscherkollegen, dürften aber für Kombinatoriker recht zugänglich sein.

C. Elsholtz (Graz)

L. Lovász: Large Networks and Graph Limits. (AMS Colloquium Publications, Vol. 60.) American Mathematical Society, Providence, Rhode Island, 2012, xiv+475 S. ISBN 978-0-8218-9085-1 H/b $ 99,–, € 74,90.

Das neueste Buch des bekannten Kombinatorikers László Lovász bietet einen ausgezeichneten Einblick in die aktuellen Entwicklungen in der Graphentheorie. Große Netzwerke (Stichwort: Internet) gewinnen mehr und mehr an Bedeutung, und diese Tatsache hat auch zu neuen Ideen in der Graphentheorie geführt.

Wie der Name schon verrät, liegt das Hauptaugenmerk des Buchs auf großen Graphen und deren Limiten, es kommen aber auch diverse andere Bereiche der Graphentheorie zur Sprache, und es werden zahlreiche Querverbindungen hergestellt, etwa zur klassischen extremalen Graphentheorie.

Das Buch beginnt mit einer kurzen, informellen Einführung in das Thema. Es folgt ein Kapitel über Graphhomomorphismen und deren Eigenschaften. Das zentrale dritte Kapitel handelt von Grenzwerten dichter Graphen. Hier wird insbesondere das noch recht junge Konzept des Graphons besprochen. Die Konvergenz von Graphen mit beschränkten Knotengraden wird im vierten Kapitel erläutert. Einem kurzen Ausblick samt Appendix schließt sich ein umfangreiches Literaturverzeichnis an, das größtenteils aus Arbeiten jüngeren Datums besteht.

Bei der Lektüre dieses Buchs wird klar, wie reichhaltig das Studium großer Netzwerke ist und zu welch tiefgründigen mathematischen Konzepten es einen führen kann. Der Autor ist ein Experte auf dem Gebiet wie kaum ein zweiter, und so kann man *Large Networks and Graph Limits* Graphentheoretikern, aber auch anderen Mathematikern, die sich für die aktuellen Trends in der Graphentheorie interessieren, uneingeschränkt empfehlen.

S. Wagner (Stellenbosch)

B. Luderer: Mathe, Märkte und Millionen. Plaudereien über Finanzmathematik zum Mitdenken und Mitrechnen. (Sachbuch) Springer Spektrum, 2013, x+168 S. ISBN 978-3-658-02773-5 P/b € 19,99.

Dieses Buch teilt sich inhaltlich in zwei Teile. Teil 1 mit dem Titel „Zinsen, Kurse und Renditen – klassische Finanzmathematik“ umfasst 19 Teilkapitel, die sich insbesondere den Themen Zinsen, Zinseszinsen und Fragen der klassischen finanzmathematischen Investionsrechnung widmen.

Teil 2, mit dem Titel „Produkte und Strategien – Mathematik der Finanzmärkte“, widmet sich überwiegend dem Bereich der Finanzmärkte unter Berücksichtigung der sogenannten „Greeks“, bis hin zu Problemen der kalten Progression, der

Berechnung der Einkommenssteuer, der Berechnung von Swaps und der Black-Scholes-Formel sowie anderen Fragen aus diesem Fachbereich. Jeder dieser Teile unterteilt sich weiter: Teil 1 hat 19, Teil 2 hat 23 Teilkapitel.

Diese Teilkapitel bestehen jeweils aus einer kleinen Geschichte mit einem einschlägigen, mathematisch formulierbaren Problem. Es ist im Wesentlichen eine sehr gelungene Darstellung des Bereichs der heutigen Bankpraxis und der dort verwendeten Terminologie. Ähnliches gilt für den Bereich der Finanzmärkte, wo Themenbereiche, die heute in den öffentlichen Finanzmarktmedien Themen darstellen, diskutiert und erklärt werden. Natürlich hat das Darstellungsdetail Grenzen. Über diese Grenzen kann man etwas anderer Meinung sein und sich vielleicht etwas mehr Tiefe bei manchen Formeln vorstellen. Dennoch ergibt sich der Eindruck einer sehr gelungen Auswahl aus den gewählten Bereichen.

Wenn man ins Detail sieht, wie z.B. in Kapitel 8 des ersten Teils, stellt man fest, dass gewisse Ungereimtheiten in der Terminologie bestehen, die verbesserungsfähig sind Z.B. wird auf Seite 25 in die Formel, die mit einem Stern gekennzeichnet ist, eingesetzt, wobei bei der Gleichung in der letzten Zeile der Logarithmus vergessen wurde und zudem mit i der Zinssatz p bezeichnet wird, obwohl in die Formel, wenn p den Zinssatz bezeichnet, mit i der Wert $p/100$ eingesetzt gehört; besonders für die intendierte Zielgruppe erscheint dies störend.

Es ergibt sich also der Eindruck, dass dieses Buch nochmals sorgfältig redigiert gehört. Ansonsten ist es sehr positiv zu sehen, da es den Konsumenten vor allem im 1. Teil sehr viel praktische Beispiele bietet, die ihm erlauben, Angebote, wie sie heute von den Banken dem Kunden gemacht werden, realistischer zu beurteilen.

Der zweite Teil ist bereits für eine Klientel gedacht, die sich etwas mit den Finanzmärkten beschäftigt, aber auch hier ist der Themenbereich so gewählt, dass er zum Verständnis für bekannte Medien wie z.B. dem *Wallstreet Journal* und anderen, die sich mit Finanzmärkten befassen, eine Unterstützung bietet.

W. Janko (Wien)

K. Nipp, D. Stoffer: Invariant Manifolds in Discrete and Continuous Dynamical Systems. (Tracts in Mathematics 21.) EMS, Zürich, 2013, ix+216 S. ISBN 978-3-03719-124-8 H/b € 58,–.

Without question, invariant manifolds are an indispensable and widely used tool in the geometric theory of dynamical systems. Their history can be traced back to Hadamard (graph transform) and Perron. While the approach of Perron has a functional-analytical flavor, the graph transform is strongly based on geometric ideas and intuition.

The present research monograph by Nipp and Stoffer – two experts in the field – is not only a systematic and well-written account to invariant manifolds illustrating how useful and modern Hadamard's graph transform still is; it also underlines their wide applicability. Indeed, the book is subdivided into three parts dealing

with discrete dynamical systems, finite-dimensional ODEs and various applications with a focus on numerical dynamics.

Initial point are flexible existence theorems for repulsive, attractive resp. hyperbolic invariant manifolds of Lipschitzian mappings defined on the products of Banach spaces. Subsequently their behavior under perturbation (including explicit estimates) and their smoothness based on the fiber-contraction principle is shown. Results on invariant foliations, in order to establish an asymptotic phase property, as well as a smoothness proof conclude the first part. By means of a time-T-mapping the above results are transferred to the continuous time situation of autonomous ordinary differential equations; this approach, based on the previous discrete time results, is also available for more general semiflows on infinite-dimensional state spaces. The remaining half of the book is devoted to applications within the field of dynamical systems; they begin with the classical Hadamard-Perron theorem on the existence of stable/unstable manifolds of a hyperbolic fixed point; strongly-stable manifolds (now in an ODE setting) are considered as well. One finds the interesting result (originally due to Kirchgraber) that for every linear multistep method there exists an asymptotically equivalent one-step method for the numerical solution of ODEs. The "slow" invariant manifolds of singularly perturbed differential equations are constructed, both in an attractive, as well as a hyperbolic framework. Thereafter, (stiff) Runge-Kutta methods applied to singularly perturbed problems are investigated. Results on invariant curves of perturbed harmonic oscillators, blow-up in singular perturbations and Runge-Kutta methods in differential-algebraic problems conclude the presentation.

The authors's intention is to provide easily applicable and yet largely quantitative results – they succeed in both respects. The book will be a helpful tool for researchers in the field and moreover, the third part on applications might serve as basis for an advanced seminar.

Ch. Pötzsche (Klagenfurt)

T. Tao: Topics in Random Matrix Theory. (Graduate Studies in Mathematics, Vol. 132.) American Mathematical Society, Providence, Rhode Island, 2012, x+282 S. ISBN 978-0-8218-7430-1 H/b $ 64,–.

Die Theorie der Zufallsmatrizen hat in der jüngsten Vergangenheit einen enormen Aufschwung erfahren und sich zu einem sehr umfangreichen Forschungsgebiet mit zahlreichen Querverbindungen zu anderen Bereichen der Mathematik sowie zur Physik entwickelt. Es ist daher unmöglich, den aktuellen Stand dieses Themas in seiner Gesamtheit in einem Band darzustellen. Auch der Autor des vorliegenden Buchs beschränkt sich auf einen prominenten Teilbereich der Zufallsmatrizen, nämlich auf die Spektralverteilung s genannter Ensembles von IID-Matrizen sowie von symmetrischen und Hermiteschen Wigner-Matrizen. Die Betrachtung schließt die wichtigen Klassen der Bernoulli-Ensembles und der Gaußschen Orthogonal- bzw. Unitär-Ensembles ein.

Das erste Kapitel des Buchs bietet eine knappe Wiederholung aller im Folgenden benötigten Fakten aus Wahrscheinlichkeitstheorie und Linearer Algebra. Dabei werden grundlegende Kenntnisse aus der Maßtheorie und der Spektraltheorie Hermitescher Matrizen vorausgesetzt.

Herzstück des Bands ist das zweite Kapitel mit folgenden Themen (Abschnitten): Konstruktion von Maßen; der Zentrale Grenzwertsatz; die Operatornorm von Zufallsmatrizen; das semizirkulare Gesetz; die freie Wahrscheinlichkeit; Gaußsche Ensembles; der kleinste Singulärwert; das zirkulare Gesetz. Dabei beschränken sich die ersten zwei Abschnitte bewusst auf skalare Zufallsvariablen, um die bei der Behandlung von Zufallsmatrizen benötigten Konzepte vorzubereiten. Eine methodische Besonderheit stellen die (rund 200) Übungsaufgaben dar, die in vielen Fällen weitere Resultate zum Inhalt haben. Ihr Beweis erfordert die Bereitschaft des Lesers, sich mit dem jeweils davor präsentierten Stoff aktiv auseinanderzusetzen; manchmal bieten unterstützende Hinweise eine kleine Hilfestellung. Die Mühe wird belohnt durch wachsende Vertrautheit mit Hilfsmitteln und Beweistechniken.

Das dritte Kapitel befasst sich mit Themen, die in enger Beziehung zum vorigen Kapitel stehen bzw. Anwendungen der dort entwickelten Theorie darstellen: die Dyson-Brownsche Bewegung, die Golden-Thompson-Ungleichung, der Dyson- und Airy-Kern Gaußscher Unitär-Ensembles (GUE) und die mesoskopische Struktur von GUE-Eigenwerten.

Der Autor legt mit seinem Werk einen übersichtlich gestalteten, gut lesbaren Text vor, der sich an Forscher auf dem Gebiet der Zufallsmatrizen wendet, sich aber auch gut als Grundlage für ein Seminar oder eine Spezialvorlesung eignet. Wie bereits angedeutet, konzentriert sich hier Terence Tao auf einen Teilbereich der Zufallsmatrizen. Wer sich beispielsweise näher mit invarianten Matrizenensembles oder Anwendungen in der Zahlentheorie und Physik befassen möchte, ist mit den neueren Monografien von Anderson-Guionnet-Zeitouni (2010), Forrester (2010) und Mehta (2004) gut beraten.

A. R. Kräuter (Leoben)

M. Willem: Functional Analysis. Fundamentals and Applications. (Cornerstones, Birkhäuser) Springer New York, Heidelberg, Dordrecht, London 2013, xii+287 S. ISBN 978-1-4614-7003-8 H/b € 41,64.

The present textbook is a very nice introduction to functional analysis with emphasis on applications to (elliptic) partial differential equations. In fact, the text is highly optimized to get to these applications without much ado, and hence many topics one might expect from a functional analysis course are only briefly touched our even missing at all.

After some preparatory material on continuity and convergence the text starts by introducing Lebesgue integration. This is done via the Daniell approach and hence

avoids the usual construction of the Lebesgue measure via outer measures (equivalence of this approach with the usual one is established along the way). There follow a brief chapter on Banach and Hilbert spaces, and chapters on Lebesgue spaces (including their duals), Sobolev spaces, and capacity. The book culminates in a final chapter on elliptic problems where the isoperimetric inequality, the Pólya-Szegő, and the Rayleigh-Faber-Krahn inequalities are proven. The book ends with an appendix on calculus topics and a nice epilog on the historical development of functional analysis.

So if you are happy with the selection of topics, this books gives you a streamlined and well written introduction which leads into a solid preparation for further studies in the direction of partial differential equations. It might be a bit though for self-study due to the lack of examples and text between theorems. Nevertheless I consider it a welcome contribution to the current literature in this subject.

As a minor remark let me mention that the author sometimes does not adhere to common denomination: For example, what is called normal convergence in the text is usually known as absolute convergence, and simple convergence is usually known as strong convergence.

G. Teschl (Wien)

Nachrichten der Österreichischen Mathematischen Gesellschaft

Preise der Österreichischen Mathematischen Gesellschaft

Der Förderpreis des Jahres 2014 geht an Dr. Christoph Haberl (SAP Wien). Eine Würdigung wird im Rahmen des Berichts über die Generalversammlung der ÖMG im nächsten Heft dieser Zeitschrift erscheinen.

Die Studienpreise des Jahres 2014 gehen an Frau Dr. Annegret Burtscher (Univ. Hannover, Dissertation an Univ. Wien und Univ. Pierre et Marie Curie, Paris 6, unter der Betreuung von J. Grant und Ph. LeFoch) und Frau Dr. Anna Geyer (Univ. Autònoma de Barcelona, Dissertation an der Univ. Wien unter der Betreuung von A. Constantin).

Die Österreichische Mathematische Gesellschaft gratuliert den Preisträgerinnen und Preisträgern herzlich.

START-Preise 2014 für Mathematikerinnen und Mathematiker

Am 16. Juni 2014 gab der Fonds zur Förderung der wissenschaftlichen Forschung (FWF) die Gewinner der heurigen START-Preise bekannt. Darunter sind Mathias Beiglböck (Univ. Wien, *Optimaler Transport und Robuste Finanzmathematik*) und Karin Schnass (Univ. Innsbruck, *Optimierung, Modelle & Algorithmen für Dictionary Learning*). Die ÖMG gratuliert herzlich zu diesem großen Erfolg.

Ehrendoktorat für Harald Niederreiter an der Universität Linz

An Prof. Harald Niederreiter (Johann Radon Institute for Computational and Applied Mathematics der Österreichischen Akademie der Wissenschaften in Linz) wurde am 25. Juni 2014 das Ehrendoktorat der Technischen Wissenschaften der Johannes Kepler-Universität Linz verliehen. Prof. Niederreiter ist Ehrenmitglied der ÖMG.

Schülerinnen- und Schülerpreis für herausragende Fachbereichsarbeiten in Mathematik oder Darstellender Geometrie 2014

Seit dem Jahr 2009 vergibt die ÖMG einen Preis für herausragende Fachbereichsarbeiten aus Mathematik und Darstellende Geometrie, der auch vom Bundesministerium für Unterricht, Kunst und Kultur unterstützt wird. Insgesamt wurden in diesem Jahr 17 Arbeiten aus ganz Österreich eingereicht. Es war sehr erfreulich, dass von den 17 Einreichungen 11 von Mädchen kamen. Alle Beiträge zeichneten sich durch hohes mathematisches Niveau, Qualität der Darstellung und Originalität aus. Eine von der ÖMG eingesetzte Jury wählte sechs besonders herausragende Arbeiten aus, welche im Rahmen des Fortbildungstags für Lehrerinnen und Lehrer an der Universität Wien (25. April 2014) gewürdigt wurden:

- Johanna Einsiedler (Mary Ward-Privatgymnasium und ORG St. Pölten): *Spieltheorie. Bimatrixspiele und deren praktische Anwendung* (Betreuerin: Mag. Maria Burmetler)
- Tobias Kietreiber (BG und BRG Tulln): *Riemannsche Zetafunktion* (Betreuerin: Mag. Anita Dorfmayr)
- Richard Löscher (BRG Petersgasse Graz): *Der Steinerkreis – Ein faszinierendes Werkzeug der Projektiven Geometrie* (Betreuer: Mag. David Stuhlpfarrer)
- Anna Niggas (GRg3 Wien): *Regelflächen – theoretisch, exemplarisch, visuell* (Betreuer: Dr. Gerhard Pillwein)
- Victoria Tiki (BG Horn): *Der Zufall in der Matheamtik* (Betreuerin: Mag. Christa Dell'mour)
- Kerstin Wolf (BG Rechte Kremszeile): *Modellierung von Wachstumsprozessen. Schimmelpilz als abstoßendes Beispiel* (Betreuer: MMag. Matthias Kittel)



Von links nach rechts: V. Tiki, R. Löscher, C. Dell'mour, K. Wolf, D. Stuhlpfarrer, T. Kietreiber, M. Kittl (verdeckt).

Die Preisträgerinnen und Preisträger waren eingeladen, ihre Arbeiten in einem kurzen Referat vorzustellen. Sie verstanden es hervorragend, in der sehr knapp bemessenen Zeit ihre Arbeit in übersichtlicher und komprimierter Weise zu präsentieren und bewiesen damit auch ihre Vortragskompetenz.

Die Preisverleihung (Urkunde, Buchpreis, einjährige Mitgliedschaft bei der ÖMG) erfolgte durch Hans Humenberger und Harald Rindler. Wir gratulieren allen Teilnehmerinnen und Teilnehmen ganz herzlich zu diesen hervorragenden Leistungen.

Die ÖMG wird den Preis in den kommenden Jahren weiter regelmäßig ausschreiben, jedoch ab dem Schuljahr 2014/15 für die dann eingeführten vorwissenschaftlichen Arbeiten.

Michael Oberguggenberger (Univ. Innsbruck, Vorsitzender der ÖMG) und Michael Drmota (TU Wien, Vorsitzender der Schülerpreisjury)



Von links nach rechts: A. Dorfmayr, Dekan H. Rindler, A. Niggas, M. Burmetler, J. Einsiedler, G. Pillwein, H. Humenberger.

Erratum

In dem Artikel *Die Exponentialfunktion als dynamisches System* von S. Götz und F. Hofbauer, IMN 223 (2013), 21–35, sind die Beschriftungen innerhalb der Figuren 3, 7 und 8 von der Druckerei leider nicht richtig wiedergegeben worden. Sie sind daher unten noch einmal (in einem anderen Verfahren) abgebildet.

Die Redaktion

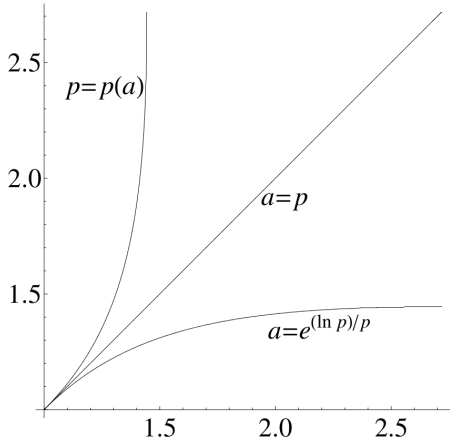


Abbildung 3: Die Graphen von $p = p(a)$ und $a = a(p)$.

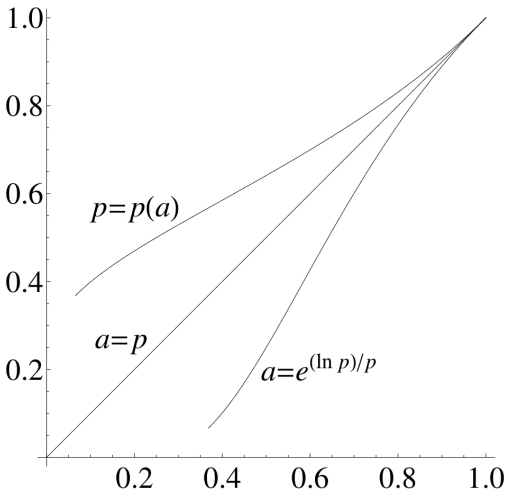


Abbildung 7: Nochmals: die Graphen von $p = p(a)$ und $a = a(p)$.

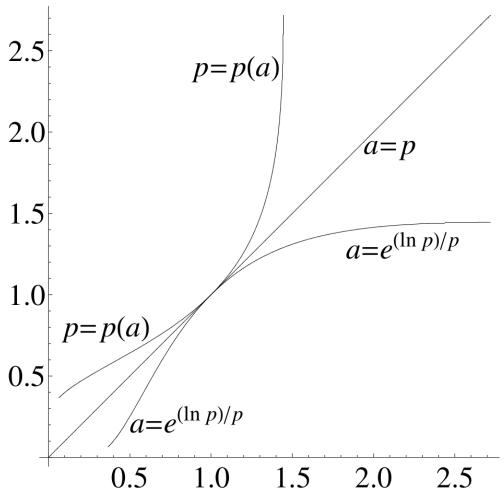


Abbildung 8: $p = p(a)$ und $a = a(p)$ auf dem maximalen Definitionsbereich.

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