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## Inhalt

Die Titelseite zeigt dessins d'enfant, die zu zu Morphismen der Riemannschen Zahlenkugel in sich gehören.

Der Begriff des dessin d'enfant, wörtlich „Kinderzeichnung", wurde von Felix Klein in seinen *Vorlesungen über das Ikosaeder* unter der Bezeichnung „Linienzug" benutzt. Seinen modernen Namen erhielt er anlässlich seiner Wiederentdeckung durch Alexander Grothendieck im Jahr 1984. Für eine holomorphe Funktion $f$ von einer Riemannschen Fläche $X$ in die Riemannsche Zahlenkugel, die nur $0, 1, \infty$ als kritische Werte besitzt (Belyi-Morphismus), ist die dazugehörige Kinderzeichnung definiert als derjenige in $X$ eingebettete bipartite Graph, dessen schwarze Knoten bzw. weisse Knoten bzw. Kanten insgesamt die Mengen $f^{-1}(0)$ bzw. $f^{-1}(1)$ bzw. $f^{-1}([0,1])$ formen. Es gibt eine Bijektion zwischen Klassen von kombinatorisch-isomorphen Zeichnungen einerseits, und Klassen von isomorphen Belyi-Morphismen andererseits.

*Literatur:* Leonardo Zapponi. What is a Dessin d'Enfant? *Notices AMS* 50 (2003), 788–789.

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# A Personal Introduction to Theoretical Dictionary Learning

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*When I was asked to write an introduction to my research area dictionary learning I was excited and said yes. Then I remembered that there is already a very readable review paper doing exactly that, namely literature reference [41]. Since I could not do it better I decided to do it differently.*

## 1 Sparsity and Dictionaries

I started to get interested in dictionary learning in 2007 at the end of the 2nd year of my Ph.D. My Ph.D. topic was roughly sparsity and dictionaries, as this was what Pierre Vandergheynst, my advisor, made almost all the group do to some degree. Since the group was a happy mix of computer scientists, electric engineers and mathematicians led by a theoretical physicist, a dictionary $\Phi$ was defined as a collection of $K$ unit norm vectors $\phi_k \in \mathbb{R}^d$ called atoms. The atoms were stacked as columns in a matrix, which by abuse of notation was also referred to as the dictionary, that is $\Phi = (\phi_1, \dots, \phi_K) \in \mathbb{R}^{d\times K}$. A signal $y \in \mathbb{R}^d$ was called sparse in a dictionary $\Phi$ if up to a small approximation error or noise it could be represented as linear combination of a small (sparse) number of dictionary atoms,

$$y = \sum_{k\in I} \phi_k x_k + \eta = \Phi_I x_I + \eta \quad \text{or} \quad y = \Phi x + \eta \quad \text{with} \quad \|x\|_0 = |I| = S, \qquad (1)$$

where $\|\cdot\|_0$ counts the non zero components of a vector or matrix. The index set $I$ storing the non zero entries was called the support with the understanding that for the sparsity level $S = |I|$ we have $S \ll d \leq K$ and that $\|\eta\|_2 \ll \|y\|_2$ or even better $\eta = 0$. Complications like infinite dimensions were better left alone since already the finite dimensional setting led to enough problems. The foremost problem being that as soon as the number of atoms exceeds the dimension, $K > d$, finding

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the best $S$-sparse approximation to a signal becomes NP-hard [8]. And while having an $S$-sparse approximation is useful for storing signals – store $S$ values and $S$ addresses instead of $d$ values – or for denoising signals – throw away $\eta$ – looking through $\binom{K}{S}$ possible index sets to find this best $S$-sparse approximation is certainly not practical. Thus people were using suboptimal but faster approximation routines and the pet routines used in the group were (Orthogonal) Matching Pursuit [32, 37, 9] and the Basis Pursuit Principle [15, 11]. Matching Pursuits are greedy algorithms, which iteratively try to construct a best S-term approximation. So given a signal $y$, initialise $a = 0$, $r = y$, $I = \emptyset$ and then for $S$ steps do:

- Find $i = \text{argmax}_k |\langle r, \phi_k \rangle|$.
- Update the support, the approximation and the residual as

$$\begin{aligned} &I = I \cup i, \\ &a = a + \langle r, \phi_k \rangle \phi_k \text{ (MP)} \quad \text{resp.} \quad a = \Phi_I \Phi_I^\dagger y \text{ (OMP)} \\ &r = y - a. \end{aligned}$$

The Basis Pursuit Principle (BP) on the other hand is a convex relaxation technique. Assuming that an $S$-sparse representation of $y$ exists, instead of solving the non-convex optimisation problem

$$(P_0) \qquad \min \|x\|_0 \quad \text{such that} \quad y = \Phi x \tag{2}$$

one solves the relaxed, convex problem

$$(P_1) \qquad \min \|x\|_1 \quad \text{such that} \quad y = \Phi x \tag{3}$$

and hopes that the solutions coincide. The relaxed problem further has the advantage that even if $y$ is contaminated by noise the solution $\hat{x}$ will be sparse in the sense that its $S$ largest (in absolute) components will provide a good sparse approximation to $y$. The big questions were: when is an $S$-sparse representation/ approximation unique and under which conditions would a suboptimal routine be able to recover it. Since for a dictionary being an orthogonal basis the answer to both problems was 'always', the first answers for a more general overcomplete dictionary (with $K > d$) were based on the concept that the dictionary was almost orthogonal. So if the largest inner product between two different atoms, called coherence $\mu = \max_{k,j:k\neq j} \left|\langle \phi_k, \phi_j \rangle\right|$, was small, sparse representations with $S \leq (2\mu)^{-1}$ were shown to be unique and both greedy and convex relaxation methods would work [49]. For a flavour of how things look like in more general, infinite settings a good starting point is [47]. Unfortunately this bound meant that in order for the best sparse approximation to be recoverable the dictionary had to be very incoherent (this incoherence being limited by the Welch bound according to which $\mu^2 \geq \frac{K-d}{d(K-1)} \approx \frac{1}{d}$) or the signal had to be very sparse, $S \lesssim \sqrt{d}$. Since in practice both schemes seemed to work fine for relatively coherent dictionaries resp. much larger sparsity levels, the coherence bound for sparse recovery was generally regarded as pessimistic and people (by now including me) were hunting for ways to go around it [20, 49, 45]. One breakthrough was in 2006, when J. Tropp could show that on average BP would be successful for sparsity levels $S \lesssim \mu^{-2}$ [50]. Following the rule that what works for BP should also work for (O)MP, I tried to prove the analogue result for (O)MP, failing horribly, but at least coming up with average case results for thresholding [44], and together with Rémi Gribonval, Holger Rauhut and Pierre Vandergheynst average case results for multichannel OMP [21].

However 2006 was foremost the year when compressed sensing started to be all the rage [10, 7], with the *Restricted Isometry Property* (RIP) being undoubtedly one of the most elegant ways to go around the coherence bound. A compressed sensing matrix $\Phi$ is said to have the RIP with isometry constant $\delta_S$ if for all subsets $I$ of size S and all coefficient sequences $x$ one has,

$$(1-\delta_S)\|x\|_2^2 \leq \|\Phi_I x\|_2^2 \leq (1+\delta_S)\|x\|_2^2, \tag{4}$$

or in other words if the spectrum of $\Phi_I^\star \Phi_I$ is included in $[1-\delta_S, 1+\delta_S]$. Note that contrary to a dictionary a compressed sensing matrix does not need to have normalised columns but if it has the RIP its column norms will be bounded by $\sqrt{1 \pm \delta_S}$. The RIP turned out to be the magic ingredient, based on which one could prove that both greedy methods [35, 36] and convex relaxation schemes [7] could recover a sparse representation, that is $x$ from $y = \Phi x$, and which was possible to have as long as $S \log S \lesssim d$. The drawback was that the only matrices one could be reasonably sure to have the RIP property in the regime $S \log S \approx d$, where based on random constructions. For a deterministic matrix the only feasible way to ensure it having RIP was to use a coherence based bound such as $\delta_S \leq (S-1)\mu$, which brought you back to square number one [48]. Still almost everybody who had been doing sparse approximation before happily turned to the investigation of compressed sensing, such as extension to signals sparse in a general (orthogonal) basis, that is recover $Bx$ from $y = \Phi Bx$ for a given basis $B$, design of matrices with RIP or recovery algorithms. And in line with the trend Holger, Pierre and me had a look at how compressed sensing would work for signals that are sparse in an overcomplete dictionary [40]. We also tried but failed to prove that OMP would work if the sensing matrix had the RIP, as it later turned out with good reason [39].

Still after 1.5 years of working on sparse recovery, compressed sensing – where you were free to choose the dictionary/sensing matrix – seemed like cheating. And weren't people forgetting that in order for compressed sensing to be applicable you first needed a dictionary to provide sparse representations? So I started to get interested in dictionary learning.

# 2 Dictionary Learning

The goal of dictionary learning is to find a dictionary that will sparsely represent a class of signals, meaning given a set of signals $y_n$, which honouring the tradition are stored as columns in a matrix $Y = (y_1 \dots y_N)$, we want to find a dictionary $\Phi$ and a coefficient matrix $X = (x_1 \dots x_N)$ such that

$$Y = \Phi X \quad \text{and} \quad X \text{ sparse.} \tag{5}$$

The 1996 paper by Olshausen and Field [13], where a dictionary is learned on patches of natural images, is widely regarded as the mother contribution to dictionary learning, but of course since I had started my Ph.D. in 2005 I was ignorant of most things having happened before 2004 [12, 31, 30, 29], and so the first dictionary learning algorithm I encountered was K-SVD in 2006 [3]. K-SVD is an alternating minimisation algorithm, which tries to solve the problem

$$(Q_2) \qquad \min \|Y - \Phi X\|_F \quad \text{such that} \quad \|x_n\|_0 \leq S \quad \text{and} \quad \Phi \in \mathcal{D}, \tag{6}$$

where $\mathcal{D}$ is defined as $\mathcal{D} = \{\Phi = (\phi_1, \dots, \phi_K) : \|\phi_k\|_2 = 1\}$, by alternating between sparsely approximating the signals in the current version of the dictionary and updating the dictionary. In particular given the current version of the dictionary $\Psi$ and training signals $y_n$ in each iteration it does the following:

- For all $n$ try to find $\min \|y_n - \Phi x_n\|_2$ such that $\|x_n\|_0 \leq S$ using (O)MP/BP to update $X$.
- For all $k$ construct the residual matrix $E_k$, by concatenating as its columns the residuals $r_n = y_n - \sum_{j \neq k} x_n(j)\psi_j$ for all $n$ where $x_n(k) \neq 0$.
- Update the $k$-th atom to be the left singular vector associated to the largest singular value of $E_k$. (Optionally update the coefficients $x_n(k) \neq 0$ in $X$.)

K-SVD worked like a charm for all sensible setups I could imagine and all signal sizes my computer could handle. Still I thought that there should be a simpler way. Unfortunately all my efforts to explain to *Matlab* how to find dictionaries in a simpler way failed and beginning of 2007 I asked Pierre for permission to finish the Ph.D. early on grounds of “never ever being able to find anything useful again”. The request was denied with a motivating speech and a “Karin, go back to your office” and a couple of weeks later I was on my way to Rémi Gribonval in Rennes, which in March felt a lot like Siberia.

Also Rémi had started to become interested in dictionary learning and since K-SVD seemed like the greedy Matching Pursuit type of way, the obvious thing to do was to try the Basis Pursuit way. So starting with the naive (unstable, intractable, nightmarish) optimisation problem

$$(R_0) \qquad \min \|X\|_0 \quad \text{such that} \quad Y = \Phi X \quad \text{and} \quad \Phi \in \mathcal{D}, \tag{7}$$

we replaced the zeros with ones to get

$$(R_1) \qquad \min \|X\|_1 \quad \text{such that} \quad Y = \Phi X \quad \text{and} \quad \Phi \in \mathcal{D}, \tag{8}$$

where $\|X\|_1 := \sum_n \|x_n\|_1$, to get a stable but unfortunately not a convex optimisation problem. Indeed while $(R_1)$ is definitely the more tractable problem, since the objective function is continuous and piecewise linear, unlike $(P_1)$ it is not convex because the constraint manifold $\mathcal{D}$ is not convex. Also it is easy to see that the problem is invariant under sign changes and permutations of the dictionary atoms, meaning that for every local minimum there are $2^K K! - 1$ other equivalent local minima.

Faced with our creation, Rémi and I asked ourselves, shall we implement it or analyse it? We decided to analyse it because it seemed easier. The question we wanted to answer was the following: Assume that we have a dictionary $\Phi_0$ and sparse coefficients $X_0$. Under which conditions is there a local minimum of $(R_1)$ at the pair $(\Phi_0, X_0)$ or in other words when can you identify the dictionary as local minimum of $(R_1)$? The next few days we spent in the seminar room, Rémi calculating the tangent space to the constraint manifold and finding a first order formulation of the problem and me pointing out in detail where it would go wrong and how it was hopeless in general. I left Rennes after two weeks and spent the rest of spring and summer going on an unreasonable amount of holidays, forgetting all about dictionary learning.

However, in autumn Rémi sent Pierre and me an email with a paper draft called dictionary identifiability, that contained geometric conditions when a basis-coefficient pair would constitute a local minimum of $(R_1)$ and our three names on it. Pierre responded fast as lightening saying, I did not contribute at all, I should not be on the paper, and after being honest with myself I tried to do the same thing, but Rémi replied, no, I'd like to keep you on it, and submitted it to a conference [23].

Reading the paper I learned that dictionary learning was also called sparse coding, that the $\ell_1$ approach was not new [52, 38], and that the field of dictionary identification had another origin in the blind source separation community. There the dictionary atoms were called the sources, the coefficients the mixing matrix and the signals the mixtures. Also one of the first theoretical insights into dictionary identification, that is, how well can you identify your sources from the mixtures, apparently came from this community [18, 4].

The part of the paper I liked most was the short sketch how to turn the rather technical, deterministic result into a simple result by assuming that the training signals were following a random sparse model. Feeling that I still owed my contribution, I set myself to work and we started digging through concentration of measure results and assembling them to make the sketch precise, succeeding first for orthogonal bases [22], and finally for general bases [24]. We did not succeed in extending our probabilistic analysis to overcomplete dictionaries, but managed to write a summary of all our results [25], so in March 2009 I could defend my thesis including a Chapter 7 on dictionary identification.

After that I did not think about dictionary identification for a while, firstly because at my new postdoc job Massimo Fornasier was paying me to think about identifying ridge functions from point queries [14], secondly because my daughter would object highly to the idea that mum would spend any time away from home not being her personal slave, and thirdly because I was trying to prove, once again, average case results for OMP, using decaying coefficients as additional assumption. This time I failed slightly more gracefully in the sense that in some highly unrealistic setting there would have been a result.

Still, failing paid off because at the end of 2010 I had an idea about dictionary learning based on the decaying coefficient assumption. If all atoms in the dictionary were equally likely to give the strongest contribution to a signal, then a very simple way to recover the dictionary should be through the maximisation program

$$(\tilde{Q}_2) \qquad \max_{\Phi} \sum_n \|\Phi^\star y_n\|_\infty^2. \tag{9}$$

Following the approach that had proven successful for $(R_1)$ I started to find the first order formulation of the problem based on the tangent space and found out that if there was a maximum at the original dictionary, it had to be a second order maximum. As the sophisticated method had failed I resorted to a brute force attack, assuming that the signals were generated from an orthonormal basis and decaying coefficients with random signs, and got a first result. Very excited, I told Rémi at the SMALL workshop about the idea and after listening patiently he said, mmmh that sounds a lot like K-SVD. Had I reinvented the wheel? No, I had a first theoretical result, showing that wheels rolled.

However, this was not the most exciting part of the SMALL workshop. The most exciting part was John Wright's talk on how to extend $\ell_1$-based dictionary identification to overcomplete dictionaries [17] and his personal confirmation that implementation of a descent algorithm was hard (in somewhat more colourful words) [16]. I was motivated to do dictionary learning again and since it was also time to look for a new job, I invented a project on dictionary learning, and then hoped for one year that someone would agree to fund it. In the meantime I followed a higher calling as personal slave to now two children.

Thus I missed the development of an interesting line of results on the sample complexity of dictionary learning [33, 51, 34, 19]. These results characterise how well the sparse approximation performance of a dictionary (for example a learned one but also a designed one) on a finite sample of training data extrapolates to future data. I also missed the development of ER-SpUD, the first algorithm which could be proven to globally recover a basis [46], and the extension of $\ell_1$-based local dictionary identification to noisy data [26, 27].

Luckily in May 2012 the project was accepted, so I could not only continue to analyse $(\tilde{Q}_2)$ but also extend it and uncover the relation to K-SVD or rather to $(Q_2)$, so that in early 2013 I had theoretical results indicating why K-SVD worked [43]. However, the most interesting development of 2013 was that two research groups independently derived algorithms, that could be proven to globally recover an overcomplete dictionary from random sparse signals [6, 2]. Their similar approach was based on finding overlapping clusters, each corresponding to one atom, in a graph derived from the correlation matrix $Y^\star Y$ and as such radically different from the optimisation based approaches, that had led to all previous results. One group then proved local convergence properties for an alternating minimisation algorithm, which alternates between sparsely approximating the signals in the current version of the dictionary and updating the dictionary, similar to K-SVD [1], while the other group tried to break the coherence barrier [5].

Indeed all results mentioned so far were valid at best for sparsity levels $S \leq O(\mu^{-1})$ or under a RIP-condition on the dictionary to be recovered, meaning under the same conditions that sparse recovery was guaranteed to work. This was somewhat frustrating in view of the fact that both BP or thresholding would on average work well for sparsity levels $S \leq O(\mu^{-2})$ [50, 44] and that in dictionary learning one usually faces a lot of average signals. So I was quite proud to scratch the coherence barrier by showing that locally dictionary identification is stable for sparsity levels up to $S \leq O(\mu^{-2})$, [42].

Now at the beginning of 2015, looking at the handful of dictionary identification results so far, it is interesting to see the two origins – sparse approximation and blind source separation – represented by the two types of results, based on optimisation on one hand and on graph clustering algorithms on the other hand. Comparing the quality of the results in terms of sample complexity, sparsity level and computational complexity is difficult, see [27] for a good attempt, as they all rely on different signal models, and it is hard to entangle the strength of the signal model – (no) noise, (no) outliers, (in)exactly sparse – from the strength of the approach. One attempt at understanding the sample complexity based solely on the signal model from an information theoretic point of view can be found in [28]. Still so far graph based algorithms are the only methods with global guarantees while optimisation schemes seem to be locally more robust to noise, outliers and coherence. In short this means that I will not get bored with dictionary learning for a while as there is plenty of work to be done, for instance trying to marry globality with robustness, deriving blind learning schemes that decide the sparsity level and dictionary size for themselves or extending the results to more realistic signal models. Moreover it is high time to take another shot at average case results for OMP.

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# Martin Hairer's regularity structures

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## 1 Introduction

Martin Hairer received the Fields medal at the ICM in Seoul 2014 for "his outstanding contributions to the theory of stochastic partial differential equations" (quoted from the ICM web page), in particular for the creation of the theory of regularity structures. Martin was born 1975 into an Austrian family living in Switzerland: his father, Ernst Hairer, is a well-known mathematician in numerical analysis working at the University of Geneva. Martin's mother has worked as a teacher in elementary school and in a Ludothek, his sister works in medical management, and his brother teaches sports. Martin completed his PhD at Geneva university under supervision of Jean-Pierre Eckmann in 2001. He is Regius Professor of mathematics at the University of Warwick, having previously held a position at the Courant Institute of New York University. He is married to the mathematician Xue-Mei Li, who also works at University of Warwick. Martin develops quite successfully the audio editor software "Amadeus", which silently reveals the Austrian background.

Martin Hairer's work on the solution of the KPZ equation and on regularity structures is astonishing by its self-contained character, its crystal-clear exposition and its far-reaching conclusions. I have rarely read research papers where a new theory is built in a such convincing and lucid way.

The purpose of this article is to explain some elements of Martin Hairer's work on regularity structures and some aspects of my personal view on it. I am not able

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to appreciate or even to describe the history, meaning and value of all problems on stochastic partial differential equations, which can be solved with the theory of regularity structures, but I do believe that there are still many future applications, for instance in Mathematical Finance or Economics, to come.

# 2 Systems and noise

Loosely speaking there are two reasons to include random influences into deterministic descriptions of a system's time evolution: either there are absolute sources of noise related to fundamental laws of physics, which need to be considered for a full description of a system, or there are subjective sources of noise due to a fundamental lack of information, which in turn can be modeled as random influences on the system. In both cases the irregularity of the concrete noise can lead to quite ill-posed equations even though it is often clear, for instance through numerical experiments, that the corresponding equations are reasonable. This can already be seen when simple stochastic systems like

$$\frac{d}{dt}S_t = S_t \dot{W}_t \, , S_0 > 0$$

are considered. This is a linear growth equation, where the growth rate is white noise $\dot{W}_t$, e.g. (independent) Gaussian shocks with vanishing expectation and covariance $\delta(t-s)$. Simulation of this equation is simple but to understand the formula analytically already needs Ito's theory. Even though white noise $\dot{W}_t$ is only defined in the distributional sense, already its integral with respect to time is a (Hölder) continuous function, which gives hope for the previous equation to have a reasonable interpretation, if one is able to define integrals along Brownian motion.

There are two main approaches towards such noisy systems: deterministic approaches which consider any realization of noise as an additional deterministic input into the system, or stochastic approaches which consider a realization of a noise as stochastic input into the system. Rough path theory and regularity structures belong to the first approach, stochastic analysis constitutes the second one.

The problem with (white) noises – by its very nature – is its persistent irregularity or roughness. Let us consider again the simplest example of noise: white noise, or its integrated version, Brownian motion. Taking a more physical point of view white noise models velocity of a Brownian particle

$$W_t := \int_0^t \dot{W}_s ds \, ,$$

i.e., a particle moving on continuous trajectories with independent increments being random variables with vanishing expectation and variance proportional to time. Simulation is easy, but apparently Brownian motion due to the independent nature of its increments is a complicated mathematical object whose existence is already an involved mathematical theorem. Brownian motion is on the other hand an extremely important tool for modeling random phenomena. For instance many important models in Mathematical Finance, where independent Gaussian shocks are a modeling assumption, are driven by Brownian motions, for instance the Black-Merton-Scholes model, the Heston model, the SABR model, etc.

Let us consider, e.g., the length of a Brownian trajectory to understand one crucial aspect of irregularity. Consider first quadratic variation which is approximated by

$$\sum_{i=0}^{2^n} \left(W_{ti/2^n} - W_{t(i-1)/2^n}\right)^2 .$$

By arguments going back to Paul Lévy it is clear that the previous sum converges almost surely to $t$, which means in turn that almost surely Brownian motion has infinite length since any continuous curve with finite length would have a vanishing quadratic variation. Therefore naive Lebesgue-Stieltjes integration with respect to Brownian motion is not an option, also Brownian trajectories are almost surely too rough for Young integration.

Mathematically speaking one needs an integration theory with respect to curves, which are not of finite total variation (like Brownian motion), in order to make sense of equations involving Brownian motion or white noise (like the Black-Merton-Scholes equation). Kiyoshi Ito's approach to deal with this problem is stochastic integration: by arguments from $L^2$-martingale theory a particular set of integrands, namely locally square integrable predictable ones, is singled out such that limits of Riemannian sums exist almost surely. Ito's insight allows, e.g., to solve stochastic differential equation in its integral form by fixed point arguments. For instance the Black-Merton-Scholes equation now reads

$$S_t = S_0 + \int_0^t S_s \, dW_s ,$$

where the right hand side is defined via stochastic integration and well understood in its probabilistic and analytic properties. Ito's stochastic calculus is a wonderful tool to work with, in particular in Mathematical Finance, as long as stochastic integration works well. This, however, might get problematic if one considers more general stochastic processes, which are not real-valued (as in the Black-Scholes equation) anymore.

Brownian motion appears as integral of a one-dimensional white noise, but we also can consider multivariate versions of white noises, where independent shocks in a space-time manner appear. Of course the irregularity of such noises is worse. The analogue of $W_t$, i.e., integration with respect to time, is not function valued anymore but only defined in the distributional sense. Hence nonlinear equations containing space-time white noises need to come up with a theory how to define non-linear functions of generalized functions, which is a well-known and hard problem. Such multivariate white noises appear in several important equations from physics, and even in equations of Mathematical Finance, it is important to understand nonlinear equations where such noises appear. Regularity structures provide a way to solve this problem in a surprisingly elegant way.

# 3 Regularity structures

Regularity structures have been introduced by Martin Hairer in a series of papers to provide solution concepts for Stochastic partial differential equations (SPDEs) like the Kardar-Parisi-Zhang (KPZ) equation, the $\Phi^4_3$ equation, the parabolic Anderson model, etc. These equations often came so far with excellent motivations from mathematical physics, convincing solution chunks in several regimes and surprisingly deep conclusions from those, but rarely with mathematically satisfying (dynamic) solution concepts. For a discussion of these issues we refer mainly to Martin Hairer's article [2] in *Inventiones Mathematicae*, but also to the great introductory paper [3].

Let us take for instance the $\Phi^4_3$-equation

$$\partial_t u = \Delta u - u^3 + \xi$$

on $\mathbb{R}_{\geq 0} \times \mathbb{R}^3$ (with periodic boundary conditions in space), where $u$ is a scalar function and $\xi$ is a space-time white noise. This important SPDE from quantum mechanics has a very singular additive term, namely space-time white noise. If one counts (parabolically) time twice as much as space dimension, $\xi$ is regular of order $-\frac{5}{2} - \varepsilon$, for any $\varepsilon > 0$ (we denote this by $-\frac{5}{2}^-$). This irregularity cannot be regularized by the heat kernel, which raises regularity only by 2 (due to well-known Schauder estimates). The resulting object is not a function yet and therefore any nonlinear operation on it is problematic. If, however, we consider the mild formulation of the $\Phi^4_3$ equation through convolution with the Green's function $K$

$$u = K * (\xi - u^3) + K * u_0 \,,$$

and if we hope for a Banach fixed point argument, we are faced with nonlinear operations on $K * \xi$ already after one iteration step: in particular we need an interpretation of $(K * \xi)^3$. It has been shown by Martin Hairer that under suitable, now well understood re-normalizations of the $\Phi^4_3$ equation of the type

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - u^3 + C_\varepsilon u_\varepsilon + \xi_\varepsilon \,,$$

where $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$ is a mollification of the white noise and $C_\varepsilon \to \infty$ is some constant depending on the mollification, one can formulate a solution concept:

indeed the limit $u_\varepsilon \to u$ as $\varepsilon \to 0$ exists, $u$ does not depend on the particular mollification involved, and the structure of the "infinities" is actually well described. More details can be found in Section 6 of [3] or Section 9 of [2]. The solution theory of the dynamic $\Phi^4_3$ is one convincing argument for the power and beauty of regularity structures.

The theory of regularity structures is based on a natural still ingenious split between algebraic properties of an equation and the analytic interpretation of those algebraic structures. These considerations are profoundly motivated by re-normalization theory from mathematical physics; however, the crucial point is their precise mathematical meaning and their clear structure. Another source of inspiration for regularity structures is Terry Lyons' *Rough Path* theory, see [4], which is somehow extended from curves to functions on higher dimensional spaces, an aspect which is outlined in the book [1].

The bridge between the algebraic and analytic world is done by the so called reconstruction operator $\mathcal{R}$, whose existence is a beautiful result from wavelet analysis interesting by itself. Regularity structures also come with precise numerical approximation results and are therefore very useful to establish numerical techniques. In the sequel we highlight on the cornerstones of the regularity structures without any proofs but with emphasis on meaning and ideas. Proofs are analytically involved, but due to the intriguing structure of the theory, which is briefly highlighted in the sequel, all the hours spent with [2] fly by quickly.

Let $A \subset \mathbb{R}$ be an index set, bounded from below and without accumulation point, and let $T = \bigoplus_{\alpha \in A} T_\alpha$ be a direct sum of Banach spaces $T_\alpha$ graded by $A$. Let furthermore $G$ be a group of linear operators on $T$ such that, for every $\alpha \in A$, every $\Gamma \in G$, and every $\tau \in T_\alpha$, one has $\Gamma\tau - \tau \in \bigoplus_{\beta<\alpha} T_\beta$.

The triple $\mathcal{T} = (A, T, G)$ is called a *regularity structure* with *model space* $T$ and *structure group* $G$. Given $\tau \in T$, we shall write $\|\tau\|_\alpha$ for the norm of its $T_\alpha$-projection.

What about the meaning of elements of $T$? they represent expansions of "functions" at some space-time point in terms of "model functions" of regularity $\alpha$, namely elements of $T_\alpha$. In order to make this precise Martin Hairer introduces "models", which do nothing else than mapping an abstract expansion to a generalized function (respecting Hölder regularity orders) for each point in space time.

Let us be a bit more precise on that in the sequel: given a test function $\phi$ on $\mathbb{R}^d$, we write $\phi^\lambda_x$ as a shorthand for the re-scaled function

$$\phi^\lambda_x(y) = \lambda^{-d}\phi\big(\lambda^{-1}(y-x)\big) .$$

For $r > 0$ we denote by $B_r$ the set of all functions $\phi : \mathbb{R}^d \to \mathbb{R}$ with $\phi \in C^r$, its norm $\|\phi\|_{C^r} \leq 1$ and supported in the unit ball around the origin. At this point we can say what we mean by *regularity of order* $\alpha < 0$ of a distribution $\eta$, namely that there exists a constant $C$ such that the inequality

$$\left|\eta(\phi_x^\lambda)\right| \leq C\lambda^\alpha$$

holds uniformly over $\phi \in B_r$, $x \in K$ and $\lambda \in ]0,1]$. Regularity of order $\alpha \geq 0$ will just be Hölder regularity.

Given a regularity structure $\mathcal{T}$ and an integer $d \geq 1$, a *model* for $\mathcal{T}$ on $\mathbb{R}^d$ consists of maps

$$\begin{aligned} \Pi\colon \mathbb{R}^d &\to L\big(T, \mathcal{D}'(\mathbb{R}^d)\big) \qquad & \Gamma\colon \mathbb{R}^d \times \mathbb{R}^d &\to G \\ x &\mapsto \Pi_x & (x,y) &\mapsto \Gamma_{xy} \end{aligned}$$

such that $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ and $\Pi_x\Gamma_{xy} = \Pi_y$. Furthermore, given $r > |\inf A|$, for any compact set $K \subset \mathbb{R}^d$ and constant $\gamma > 0$, there exists a constant $C$ such that the inequalities

$$\left|\big(\Pi_x\tau\big)(\phi_x^\lambda)\right| \leq C\lambda^\alpha \|\tau\|_\alpha\,, \qquad \|\Gamma_{xy}\tau\|_\beta \leq C|x-y|^{\alpha-\beta}\|\tau\|_\alpha\,,$$

hold uniformly over $\phi \in B_r$, $(x,y) \in K$, $\lambda \in ]0,1]$, $\tau \in T_\alpha$ with $\alpha \leq \gamma$, and $\beta < \alpha$. In words: for every space-time point $x \in \mathbb{R}^d$ the distribution $\Pi_x\tau$ interprets each $\tau \in T$ accordingly. The role of the group $G$ also becomes clear at this point: $G$ is a collection of linear maps on $T$ which encode how expansions of a fixed analytic object transform when considering different space-time points.

Models interpret abstract expansions (elements of $T$) at each space time point $x$; these model functions constitute a frame (at each point in space time) on which one can construct generalized functions expressible in this frame with point-varying coordinates. Martin Hairer calls these generalized functions "modeled distributions". In particular they depend on the model $(\Pi,\Gamma)$.

Given a regularity structure $\mathcal{T}$ equipped with a model $(\Pi,\Gamma)$ over $\mathbb{R}^d$, the space $\mathcal{D}^\gamma = \mathcal{D}^\gamma(\mathcal{T},\Gamma)$ is given by the set of functions $f : \mathbb{R}^d \to \bigoplus_{\alpha<\gamma} T_\alpha$ such that, for every compact set $K$ and every $\alpha < \gamma$, there exists a constant $C$ with

$$\|f(x) - \Gamma_{xy}f(y)\|_\alpha \leq C|x-y|^{\gamma-\alpha} \tag{1}$$

uniformly over $x,y \in K$.

A priori it is not at all clear whether modeled distributions actually allow for an interpretation as distribution on space time: the most fundamental result in the theory of regularity structures then states that given a modeled distribution $f \in \mathcal{D}^\gamma$ with $\gamma > 0$, there exists a *unique* distribution $\mathcal{R}f$ on $\mathbb{R}^d$ such that, for every $x \in \mathbb{R}^d$, $\mathcal{R}f$ equals $\Pi_x f(x)$ near $x$ up to order $\gamma$. More precisely, one has the following reconstruction theorem, whose proof relies on deep results from wavelet analysis (for the beautiful proof see Martin Hairer's *Inventiones* article [2], Theorem 3.10):

Let $\mathcal{T}$ be a regularity structure and let $(\Pi,\Gamma)$ be a model for $\mathcal{T}$ on $\mathbb{R}^d$. Then, there exists a unique linear map $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{D}'(\mathbb{R}^d)$ such that

$$\left|\left(\mathcal{R}f - \Pi_x f(x)\right)(\phi_x^\lambda)\right| \lesssim \lambda^\gamma , \tag{2}$$

uniformly over $\phi \in B_r$ and $\lambda \in ]0,1]$, and locally uniformly in $x \in \mathbb{R}^d$.

In other words: modeled distributions, i.e., functions into the space of abstract expansions, come with appropriate consistency conditions (1), which still allow to construct a distribution such that the model $(\Pi,\Gamma)$ remains tangent up to given order $\gamma$ as in (2). Notice that the existence of the reconstruction operator should be interpreted as the construction of an integral

$$(\mathcal{R}f)(\phi) = \text{“}\int \sum_{\alpha\in A} f_\alpha(x)(\Pi_x\tau_\alpha)(x)\phi(x)dx\text{”} ,$$

where we write $f(x) = \sum_{\alpha\in A} f_\alpha(x)\tau_\alpha$ in a sloppy way for $f \in \mathcal{D}^\gamma$ and where we write a value for the distribution $\Pi_x\tau_\alpha$ at $x$ even though this is not possible in all cases of interest. Notice also that the existence of such integrals is highly nontrivial, since we are performing summations over very singular objects.

At this point it might become clear where regularity structures are leading us: translate a “real world equation” into an equation on abstract expansions, solve this equation and translate the solution – via the reconstruction operator back to “real world”. To realize this idea it is necessary to understand how linear equations are translated to abstract expansion spaces, which is the world of Schauder estimates. Classical Schauder estimates tell that for a kernel $K : \mathbb{R}^d \to \mathbb{R}$ being smooth everywhere, except for a singularity at the origin of approximate homogeneity $\beta - d$ for some $\beta > 0$, the integral operator $K : f \mapsto K * f$ maps $\mathcal{C}^\alpha$, i.e., $\alpha$-Hölder continuous functions, into $\mathcal{C}^{\alpha+\beta}$ for every $\alpha \in \mathbb{R}$ (except for those values for which $\alpha + \beta \in \mathbb{N}$). In the theory of regularity structures this naturally amounts to lifting the integral operator $K$ to an operator on modeled distributions $\mathcal{K} : \mathcal{D}^\gamma \to \mathcal{D}^{\gamma+\beta}$, which commutes with reconstruction, i.e., $K * \mathcal{R}f = \mathcal{R}\mathcal{K}f$ for all modeled distributions $f \in \mathcal{D}^\gamma$.

Martin Hairer needs three basic ingredients for such a type of construction:

1. in order to describe the behavior of regular (smooth in the classical sense) parts of the integral operator $K$ polynomials should be part of the given regularity structure.

2. an abstract integration operator $I : T \to T$, which encodes the action of $K$ on singular objects.

3. a compatibility of the given model, abstract integration and the to-be-lifted integral kernel $K$, i.e. between $\Pi_x I\tau$ and $K * \Pi_x\tau$ near $x$.

This leads to the introduction of admissible models where the desired lift $\mathcal{K}$ can actually be performed for certain kernels of homogeneity $\beta - d$. All proofs can be found in Section 5 of [2].

With all these ingredients we can return to the $\Phi^4_3$ equation introduced at the beginning of this section and see how regularity structures enter the stage: we can construct a tailor-made regularity structure, which is generated in a free way by variables representing $\xi$, $(K*\xi)^3$, $K*(K*\xi)^3$, etc., and by space-time polynomials. Let us replace $K$ by an abstract integration operator $I$, $\xi$ by an abstract variable $\Xi$ of order $-\frac{5}{2}^-$, and $u$ by $\Phi$, then we can – having Banach's fixed point theorem in mind – build a regularity structure generated by $\Xi$, $I(\Xi)^3$, etc. Associated orders of regularity are $-\frac{5}{2}^-$, $-\frac{3}{2}^-$, etc. There will be only finitely many generators of negative orders in a minimal model space $T$ (i.e., a Banach fixed point consideration makes sense). The $\Phi^4_3$-equation is then translated into a fixed point equation on the coefficient space $\mathcal{D}^\gamma$ with respect to this roughly described regularity structure $\mathcal{T}$, i.e.

$$\Phi = I(\Xi - \Phi^3) + \text{ polynomials},$$

where 'polynomials' denotes terms describing the 'smooth' part of the operator $\mathcal{K}$.

Apparently the equation can be solved abstractly by the very construction in some $\mathcal{D}^\gamma$ spaces if we consider models where $\Xi$ is mapped to a mollified noise, since then it is easy to define the meaning of objects like $I(\Xi)^3$. Reconstruction then yields a solution of the $\Phi^4_3$-equation with mollified noise and initial value $u_0$.

If, however, the mollified noise converges to white noise, the actual power of the regularity structures is revealed: as outlined we obtain solutions for each model $(\Pi,\Gamma)$, first in $\mathcal{D}^\gamma$, and second – via reconstruction – as distributions on space time if noise is mollified. However, the algebraic structure of $T$ prescribes precisely which "products" of distributions have to be defined appropriately, which in turn means to construct models even for singular noises. This procedure involves re-normalizations, i.e., the real world equation being satisfied after reconstruction changes. Notice that re-normalizing products has to be done only for finitely many elements of $T$ of regularity order less than zero (in this sub-critical situation) to guarantee a well-defined model, well-defined modeled distributions and well-defined reconstruction. Re-normalization groups, whose dimension depends on the particular situation, govern the structure of different models which can be defined as limits of models with mollified noises. Hence the re-normalization of the real world equation is transferred to re-normalization of models, which can be analyzed by methods from group theory and algebra. The corresponding solution concept of the SPDE does not depend on the chosen mollification and satisfies all necessary requirements, hence deserves to be called *the* solution of the $\Phi^4_3$-equation.

# 4 Ito calculus, rough paths and Regularity Structures

Let us consider two natural examples of regularity structures, which explain the following discussion:

## 4.1 The polynomial regularity structure

The most classical example of a regularity structure is the polynomial one: the abstract expansions are abstract polynomials in $d$ variables $X^1,\dots,X^d$, models are concrete polynomials, $A=\mathbb{N}$ and we can identify $G$ with the group of translations acting on $\mathbb{R}[X^1,\dots,X^d]$ via $\Gamma_h p := p(.-h)$ for $h\in\mathbb{R}^d$.
The canonical polynomial model is then given by

$$\big(\Pi_x X^k\big)(y) = (y-x)^k\ , \qquad \Gamma_{xy} = \Gamma_{y-x}\ . \tag{3}$$

If we choose the canonical polynomial model, then the space of modeled distributions $\mathcal{D}^\gamma$ corresponds to the space of Hölder continuous functions $\mathcal{C}^\gamma$ (with proper understanding for integer $\gamma$). In other words the canonical regularity structure on $\mathbb{R}^d$ speaks about Hölder functions and their local Taylor expansions (see Section 3 of [3]).

## 4.2 A regularity structure for rough paths

Given $\alpha\in(\frac{1}{3},\frac{1}{2})$ and $n>1$. Define $A=\{\alpha-1,2\alpha-1,0,\alpha\}$. We consider a free vector space $T$ generated by $n$ order $\alpha$ elements $W_j$, $n$ order $\alpha-1$ elements $\Xi_j$, and $n^2$ order $2\alpha-1$ elements $W_j\Xi_i$, and one order 0 element 1.
We choose $G=\mathbb{R}^n$ and define the action on $T$ via

$$\Gamma_x 1 = 1\ , \qquad \Gamma_x\Xi_i = \Xi_i\ , \qquad \Gamma_x W_i = W_i - x^i\ , \qquad \Gamma_x(W_j\Xi_i) = W_j\Xi_i - x^j\Xi_i\ .$$

This is a regularity structure on $\mathbb{R}$ and models for this regularity structure are precisely rough paths of order $\alpha$ with values in $\mathbb{R}^n$. Let us be more precise on this. Take a model $(\Pi,\Gamma)$, then the formulas

$$\big(\Pi_s 1\big)(t) = 1\ , \qquad \big(\Pi_s W_j\big)(t) = X^j_t - X^j_s$$
$$\big(\Pi_s\Xi_j\big)(\psi) = \int \psi(t)\, dX^j_t\ , \qquad \big(\Pi_s W_j\Xi_i\big)(\psi) = \int \psi(t)\, d\mathbb{X}^{i,j}_{s,t}\ ,$$

for test functions $\psi$ define a (geometric) rough path $(s,t)\mapsto (X^j_t - X^j_s, \mathbb{X}^{i,j}_{s,t})$, furthermore $\Gamma_{su}=\Gamma_{X_u-X_s}$. Apparently $X$ only needs to be Hölder continuous of order $\alpha$, whereas $\mathbb{X}$ satisfies a sort of Hölder condition of order $2\alpha$. The algebraic relationships for models translate to the relationships for rough paths.

Modeled distributions appear in this setting as natural integrands with respect to rough paths: take $Y\Xi_j + Y_i' W_i \Xi_j \in \mathcal{D}^{3\alpha-1}$, then reconstruction actually defines a curve, which can be seen as the integral $\int Y dX_j$ (see Section 3 of [3]).

At this point it is clear that the split between algebraic structures, which depend on the concrete SPDE, and their analytic reconstruction is crucial for the flexibility on introducing re-normalizations. In contrast Ito's stochastic calculus does work differently: stochastic integration is introduced via martingale arguments, so the problem that the increment of Brownian motion on an interval of length $\Delta t$ only scales like $\sqrt{\Delta t}$ is cured by predictability of integrands. No additional term of order $\Delta t$ is introduced to replace the missing order. More precisely: locally in time the stochastic integral $\int_0^t h_s \, dW_s$ looks like $h_s \Delta W_s$, hence miraculously Riemannian sums converge even though the local expansion is only given up to order $\sqrt{\Delta t}$.

Rough path theory, such as regularity theory, argues that an integral along a path with low regularity, like a trajectory of Brownian motion, can only be defined if an additional term of order $\Delta t$ is introduced describing the integral locally up to order $\Delta t$. The set of integrands changes with respect to Ito integration: for stochastic integration every locally square integrable predictable integrand is eligible, whereas in case of rough path theory predictability is replaced by several analytic conditions (in particular also anticipative integrands are possible).

Ito's approach has the advantage of a robust large set of integrands, namely all bounded predictable processes, which work for a large set of integrators, namely all semi-martingales. Not only Brownian motion is a possible integrator but also Lévy processes, or more general jump processes. On the other hand regularity of the stochastic integral is low: usually a stochastic integral depends only in a measurable way on the integrator and not better.

Rough path theory or the theory of regularity structures in contrast has the advantage of a considerably more regular dependence of the integral on the integrator. The price to pay is a less robust, more regular set of integrands. To be precise here, the regularity is described in terms of regularities of the reconstruction operator, which depends in a continuous way on the model. This amazing fact means for instance in case of the SABR model (a particular stochastic volatility model)

$$dX_t = X_t Y_t \, dW_t^1 \, , \; dY_t = \sigma Y_t \, dW_t^2 \, , \; X_0 \geq 0 \, , \; Y_0 \geq 0 \, ,$$

that the solution process $(X, Y)$ only depends in a measurable way on the Brownian input paths $(W^1, W^2)$, whereas in a continuous way on Brownian motion together with its Lévy areas $(W^1, W^2, \int W^1 \, dW^2 - W^2 \, dW^1)$, which is an essential part of a model describing integration with respect to Brownian motion in the world of regularity structures. In other words: the expected higher regularity of the solution map of a stochastic differential equation is discovered by introducing stochastic integration in a deterministic way. This important insight is the content of Terry Lyons' Universal Limit Theorem of rough path theory, which appears now as one particular case of a regularity structure, see for instance [4].

It remains one interesting topic for future research to describe settings where both somehow complementary approaches deal with rough objects, namely stochastic integration and the theory of regularity structures, are combined.

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# On the variety of Euclidean point sets

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## 1. Unions of intervals

Let $\mathcal{U}$ be the family of all open subsets of the Euclidean number line $\mathbb{R}$. Motivated by the fact that each open subset of $\mathbb{R}$ is a union of mutually disjoint open intervals, we define a family $\mathcal{A}$ of point sets so that $X \in \mathcal{A}$ if and only if $X$ is a closed subset of $\mathbb{R}$ which is a union of mutually disjoint nondegenerate closed intervals. In other words, the members of $\mathcal{A}$ are precisely the closed subspaces of $\mathbb{R}$ where no component of any space is a singleton.

It is common opinion that the topological structure of an arbitrary closed subset of $\mathbb{R}$ may be more complicated than of any open subset of $\mathbb{R}$, although grounds for this opinion are rather informal. Our first goal is to support this view by pointing out that the structural discrepancy in question is revealed by a clear cardinal discrepancy already when only point sets in the family $\mathcal{U} \cup \mathcal{A}$ are considered. In fact, there are more topological types of members of $\mathcal{A}$ than of members of $\mathcal{U}$!

Naturally, both families $\mathcal{U}$ and $\mathcal{A}$ have the cardinality $c$ of the continuum $\mathbb{R}$. But the family $\mathcal{U}$ contains only countably many topologically distinct members. Indeed, since each $U \in \mathcal{U}$ can be written as a union of countably many mutually disjoint open intervals, if $\emptyset \neq U \in \mathcal{U}$ then there is precisely one $n \in \mathbb{N} \cup \{\infty\}$ so that $U$ and $\bigcup_{k=1}^{n} ]k, k+1[$ are homeomorphic subspaces of $\mathbb{R}$. (In order to avoid potential misinterpretations, $0 \notin \mathbb{N}$, i.e., $\mathbb{N} = \{1, 2, 3, \ldots\}$.) In particular, each open subspace of $\mathbb{R}$ with infinitely many components is homeomorphic to $\mathbb{R} \setminus \mathbb{Z}$. On the other hand, the following theorem shows that there are $c$ topologically distinct point sets in the family $\mathcal{A}$.

**Theorem 1.** *There are $c$ mutually non-homeomorphic compact subspaces of $\mathbb{R}$ without singleton components.*

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The situation is different when, instead of topological types of point sets in $\mathbb{R}$, *metrical* types are considered, which means that *continuity* is sharpened to *uniform continuity*. (The metric is the inherited Euclidean metric of $\mathbb{R}$.) A fortiori, topologically distinct point sets are always metrically distinct. Thus the interiors of the $c$ topologically distinct compact point sets given by Theorem 1 must be metrically distinct because every $A \in \mathcal{A}$ equipped with the Euclidean metric is a completion of the interior of $A$ equipped with the Euclidean metric. Therefore, the total number of metrical types of the open point sets in $\mathbb{R}$ is $c$ and hence greater than the total number of their topological types. (As a consequence, there exists a collection of $c$ metrically distinct and topologically similar open subsets of $\mathbb{R}$.) Certainly, metrically distinct *compact* subspaces of $\mathbb{R}$ cannot be homeomorphic. But, as the following proposition shows in an illustrative way, it is possible to track down $c$ members of $\mathcal{A}$ which are metrically distinct and topologically similar. And there is also an illustrative stack of $c$ metrically distinct and topologically similar open subsets of $\mathbb{R}$.

**Proposition 1.** *For each real number $u \geq 2$ define $X_u \in \mathcal{A}$ and $X_u^\circ \in \mathcal{U}$ via*

$$X_u := \bigcup_{n=1}^{\infty} \left[2^{u^n}, 2^{u^n} + u^n\right] \qquad \textit{and} \quad X_u^\circ = \bigcup_{n=1}^{\infty} \left]2^{u^n}, 2^{u^n} + u^n\right[.$$

*If $2 \leq v < w$ then there is no uniformly continuous bijection from $X_v$ onto $X_w$ or from $X_v^\circ$ onto $X_w^\circ$.*

Beside the topological and the metrical view there is a third natural way to look at the point sets in the families $\mathcal{U}$ and $\mathcal{A}$. Two sets $X, Y \subset \mathbb{R}$ are *order-isomorphic* if and only if there exists a strictly increasing function from $X$ onto $Y$. Of course, order-isomorphic sets $X, Y \subset \mathbb{R}$ need not be homeomorphic subspaces of $\mathbb{R}$. (Consider for example $X = [2,3]$ and $Y = \{1\} \cup ]2,3]$.) However, if $X, Y \subset \mathbb{R}$ are open or closed then the spaces $X, Y$ must be homeomorphic if the sets $X, Y$ are order-isomorphic. (For the Euclidean topology restricted to a closed or open set $S \subset \mathbb{R}$ coincides with the order topology on $S$ induced by the natural ordering of the real numbers in $S$.) In particular, topologically distinct sets in the family $\mathcal{U} \cup \mathcal{A}$ are never order-isomorphic. On the other hand, the $c$ metrically distinct open resp. closed sets in Proposition 1 are obviously order-isomorphic. It is also possible to establish a completely converse situation.

**Proposition 2.** *There are $c$ metrically (and hence topologically) similar sets in the family $\mathcal{U}$ and in the family $\mathcal{A}$, respectively, which are mutually not order-isomorphic.*

Thus, other than concerning topological types and similarly as concerning metrical types, there is no discrepancy between the total numbers of the order types of open sets and of the order types of closed sets in the number line.

*Remark.* By a classic theorem due to Mazurkiewicz und Sierpiński [4] there are precisely $\aleph_1$ compact and countable Hausdorff spaces up to homeomorphism. ($\aleph_1$ is the least cardinal number greater than the cardinality $\aleph_0$ of a countably infinite set, whence $\aleph_1 \leq c$.) As a consequence, since each countable metric space can be embedded in $\mathbb{R}$ (see [1] 4.3.H.b), the space $\mathbb{R}$ has uncountably many mutually non-homeomorphic compact subspaces. Theorem 1 is an improvement of this consequence because one cannot rule out the existence of an uncountable set whose cardinality is smaller than $c$. (Actually, the existence of such a set is the negation of Cantor's continuum hypothesis which, due to Gödel and Cohen, is independent of the Zermelo-Fraenkel axioms of set theory.)

## 2. Unions of cubes

The cardinal discrepancy between all topological types of open and all topological types of closed sets in the realm of linear point sets already vanishes in the realm of planar point sets. Indeed, the following theorem shows that for arbitrary dimensions $n \geq 2$ the Euclidean space $\mathbb{R}^n$ contains $c$ topologically distinct open point sets whose closures have no singleton components and are topologically distinct as well. (As usual, $\overline{X}$ denotes the closure of $X$.)

**Theorem 2.** *For each $n \geq 2$ there is a family $\mathcal{F}_n$ of open subsets of the Euclidean space $\mathbb{R}^n$ such that $\mathcal{F}_n$ has cardinality $c$ and neither $U,V$ nor $\overline{U},\overline{V}$ are homeomorphic subspaces of $\mathbb{R}^n$ whenever $U,V \in \mathcal{F}_n$ and $U \neq V$. Moreover, the family $\mathcal{F}_n$ can be chosen so that for every $X \in \mathcal{F}_n$ the set $\overline{X}$ is a compact union of closed cubes of the form $[a_1,a_1+h] \times \cdots \times [a_n,a_n+h]$ with $h > 0$ where the interiors of distinct cubes are always disjoint. Alternatively, the family $\mathcal{F}_n$ can be chosen so that $\overline{X}$ is a union of unit cubes $[k_1,k_1+1] \times \cdots \times [k_n,k_n+1]$ with $k_1,\ldots,k_n \in \mathbb{Z}$ for every $X \in \mathcal{F}_n$.*

It is impossible that every set $U$ in the uncountable family $\mathcal{F}_n$ is a union of mutually disjoint open cubes (or that $\overline{U}$ is a union of mutually disjoint compact cubes for every $U \in \mathcal{F}_n$.) This is because two open subspaces of $\mathbb{R}^n$ where each component is an open cube are homeomorphic if and only if the total numbers of components coincide. But if one sets a value on *disjoint* cubes, it is possible to achieve the following results.

**Proposition 3.** *For each of the $c$ sets $S \subset \mathbb{N}$ define an open set $Y_S \subset \,]-1,1[^n$ via*

$$Y_S := \,]-1,0[^n \,\cup\, \bigcup_{m=1}^{\infty} \Big( \,]2^{-2m}, 2^{-2m+1}[ \,\setminus\, \bigcup_{s\in S} \{2^{-2s} + k\tfrac{2^{-2s}}{s+1} \mid k = 1,2,\ldots,s\} \Big)^n.$$

*Obviously, all $Y_S$ are unions of infinitely many mutually disjoint open cubes and hence homeomorphic spaces, and $\overline{Y_S} = [-1,0]^n \cup \bigcup_{m=1}^{\infty} \big[2^{-2m}, 2^{-2m+1}\big]^n$ for every*

$S \subset \mathbb{N}$. *But whenever $S \neq S'$, there is no bijection $f$ from $Y_S$ onto $Y_{S'}$ such that both $f$ and $f^{-1}$ are uniformly continuous.*

**Theorem 3.** *For each dimension $n$ there exists a family $\mathcal{V}_n$ of open subsets of $\mathbb{R}^n$ such that*

- $(i)$ *$\mathcal{V}_n$ has the cardinality $c$;*
- $(ii)$ *each $V \in \mathcal{V}_n$ is a union of mutually disjoint open cubes;*
- $(iii)$ *$\overline{V}$ is a union of mutually disjoint nondegenerate compact cubes if $V \in \mathcal{V}_n$;*
- $(iv)$ *all $V \in \mathcal{V}_n$ are metrically distinct but topologically similar;*
- $(v)$ *all $\overline{V}$ ($V \in \mathcal{V}_n$) are mutually non-homeomorphic compact subspaces of $\mathbb{R}^n$.*

## 3. Proof of Theorem 1

In the following we need *Cantor derivatives* but in order to keep the story simple we use only *finite* derivatives. If $P$ is a point set in a Hausdorff space then the first derivative $P'$ of $P$ is the set of all limit points of $P$. The first derivative of any set is closed. And $P$ is closed if and only if $P' \subset P$. Further, with $P^{(0)} = P$, for every $k = 1,2,3,\dots$ the $k$-th derivative $P^{(k)}$ of $P$ is given by $P^{(k)} = (P^{k-1)})'$. Consequently, all derivatives of a closed set $A$ are closed and $A = A^{(0)} \supset A^{(1)} \supset A^{(2)} \supset A^{(3)} \supset \cdots$. (And possibly but not necessarily, $A^{(k)} = A^{(m)}$ whenever $k \geq m$ for some $m \in \mathbb{N}$.) For abbreviation let $h(x) := \frac{2}{\pi}\arctan x$. (Then $h$ is a strictly increasing function from $[0,\infty[$ onto $[0,1[$.)

In order to prove Theorem 1, we construct a compact subspace $X_S$ of $\mathbb{R}$ without point components for each infinite $S \subset \mathbb{N}$ so that $X_S, X_{S'}$ are never homeomorphic for distinct sets $S, S'$. Define for each $n \in \mathbb{N}$ a countable subset $K_n$ of the interval $[5n, 5n+1]$ in the following way. For arbitrary $X \subset [0,1]$ define $F(X) \subset [0,1]$ by

$$F(X) := h(\{n-1+x \mid n \in \mathbb{N} \wedge x \in X\}) \cup \{1\}$$

and starting with $A_1 = \{1 - \frac{1}{m} \mid m \in \mathbb{N}\} \cup \{1\}$ put $A_{n+1} = F(A_n)$ for $n = 1,2,3,\dots$ and define $K_n := \{x + 5n \mid x \in A_n\}$ for each $n \in \mathbb{N}$. Obviously, $K_n$ is a closed subset of $[5n, 5n+1]$ with $\max K_n = 5n+1$ for each $n \in \mathbb{N}$. Hence $K_n$ is always compact. Furthermore it is evident that $K_n$ is well-ordered by the natural ordering $\leq$. (Besides, one may realize that the order type of $(K_n, \leq)$ is $\omega^n + 1$.)

By construction, for each $n \in \mathbb{N}$ the $k$-th Cantor derivative $K_n^{(k)}$ is infinite whenever $k < n$ and empty whenever $k > n$ and $K_n^{(n)} = \{5n+1\}$. For each $n \in \mathbb{N}$ let $g_n$ be the reflection in the point $5n+2$, whence $g_n(x) = 10n+4-x$ for $x \in \mathbb{R}$ and $g_n([5n, 5n+1]) = [5n+3, 5n+4]$. For every $a \in K_n$ choose $0 < \varepsilon(a) \leq 1$ such that $[a, a+\varepsilon(a)] \cap K_n = \{a\}$ whenever $a \in K_n$ and put $\varepsilon(5n+1) = 1$. (For example put $\varepsilon(a) = (a'-a)/2$ where $a' = \min\{x \in K_n \mid x > a\}$ whenever $a \in K_n \setminus \{5n+1\}$.)

Finally, for each infinite set $S \subset \mathbb{N}$ define

$$X_S := h\left(\bigcup\left\{\bigcup\{[a,a+\varepsilon(a)]\cup g_n([a,a+\varepsilon(a)]) \mid a\in K_n\}\ \middle|\ n\in S\right\}\right)\ \cup\ [1,2].$$

It is plain that $X_S$ is always a closed and hence compact subset of $[0,2]$. Obviously, all components of the space $X_S$ are compact (and nondegenerate) intervals and $C_n := h([5n+1,5n+3])$ is a component of $X_S$ for every $n\in S$. Hence we can write $X_S = \bigcup\{[a_j,b_j] \mid j\in\mathbb{N}\}$ where always $a_j < b_j$ and the intervals $[a_j,b_j]$ are mutually disjoint.

Consider the point set $B_S := \{a_j \mid j\in\mathbb{N}\}\cup\{b_j \mid j\in\mathbb{N}\}$, which clearly is the boundary of the point set $X_S$ in the Euclidean space $\mathbb{R}$. The point set $B_S$ is also topologically determined within the subspace $X_S$ of $\mathbb{R}$ because $B_S$ equals the set of all points $x\in X_S$ so that for every component $C$ of the space $X_S$ the point set $C\setminus\{x\}$ remains connected in the space $X_S$. Let $\mathcal{L}_S$ be the family of all components $C$ of $X_S$ such that $C$ contains precisely two limit points of $B_S$. (Thus the family $\mathcal{L}_S$ is also topologically determined with respect to the space $X_S$.) By construction we have $\mathcal{L}_S = \{C_n \mid n\in S\}$ for each infinite $S\subset\mathbb{N}$. (Any component $C$ of $X_S$ with $C\neq C_n$ for every $n\in S$ contains at most one limit point of $B_S$.) Moreover, if $S$ is any infinite subset of $\mathbb{N}$ and if $k\in\mathbb{N}$ and $n\in S$ then $B_S^{(k)}\cap C_n=\emptyset$ when $k>n$ and $B_S^{(k)}\cap C_n\neq\emptyset$ when $k\leq n$. Consequently,

$$S = \left\{\min\left\{m\in\mathbb{N} \mid B_S^{(m+1)}\cap C=\emptyset\right\}\ \middle|\ C\in\mathcal{L}_S\right\}$$

and hence the set $S$ is completely determined by the topology of the space $X_S$. Thus for distinct infinite sets $S,S'\subset\mathbb{N}$ the spaces $X_S, X_{S'}$ cannot be homeomorphic.

*Remark.* The clue in the previous proof is to approximate certain intervals from *both* the left and the right. The proof would not work with approximations, say, from the left. Because if we consider the compact spaces

$$\tilde{X}_S := h\left(\bigcup\left\{\bigcup\{[a,a+\varepsilon(a)] \mid a\in K_n\}\ \middle|\ n\in S\right\}\right)\ \cup\ [1,2]$$

for arbitrary infinite $S\subset\mathbb{N}$ then all spaces are order-isomorphic and hence homeomorphic! (In fact, with the notation as in the proof of Proposition 2 below, for any infinite $S\subset\mathbb{N}$ the linearly ordered set $(\mathcal{C}(\tilde{X}_S),\prec)$ is well-ordered and order-isomorphic to the set of all ordinal numbers $\alpha\leq\omega^\omega$.)

## 4. Proof of Theorem 3

The proof of Theorem 1 can be adapted in order to verify Theorem 3. We replace each point set $X_S = \bigcup\{[a_j,b_j] \mid j\in\mathbb{N}\}$ with $\tilde{X}_S = \bigcup\{[a_j,b_j]^n \mid j\in\mathbb{N}\}$ and claim that these compact subspaces of $\mathbb{R}^n$ are mutually non-homeomorphic also for arbitrary dimensions $n$. As a consequence, Theorem 3 is settled by defining $\mathcal{V}_n$ as the family of all open sets $\bigcup_{j=1}^{\infty} ]a_j, b_j[^n$ corresponding to $X_S$ represented as above with $S$ running through the infinite subsets of $\mathbb{N}$. Indeed, (i), (ii), (iii) are obviously satisfied and (iv) follows from (v) since $\overline{V}$ is the completion of each metric space $V \in \mathcal{V}_n$.

Since an elimination of one point of a cube never destroys its connectedness, we cannot adopt the argumentation using the set $B_S$ in higher dimensions. But fortunately we can stay very close to the proof of dimension 1 by transforming the concept of Cantor derivatives from point sets of a topological space to *families of components* of the space in the following way.

Let $\mathcal{G}$ be the family of all components of a Hausdorff space. For every $\mathcal{F} \subset \mathcal{G}$ define $\mathcal{F}' = \mathcal{F}^{(1)} := \{G \in \mathcal{G} \mid G \cap \overline{\bigcup(\mathcal{F} \setminus \{G\})} \neq \emptyset\}$ and $\mathcal{F}^{(k+1)} := (\mathcal{F}^{(k)})'$ for every $k \in \mathbb{N}$. Now referring to $\tilde{X}_S$ let $\tilde{\mathcal{L}}_S$ be the family of all components $C \in \mathcal{G}$ such that $C \cap \overline{\tilde{X}_S \setminus C}$ contains precisely two points. Then, similarly as in the proof of Theorem 1, the set $S$ is topologically characterized via

$$S = \left\{ \min \left\{ m \in \mathbb{N} \mid C \notin \mathcal{G}^{(m+1)} \right\} \;\middle|\; C \in \tilde{\mathcal{L}}_S \right\}.$$

## 5. Proof of Theorem 2

In the following, if $X \subset \mathbb{R}^n$ then $\overline{X}$ is the closure of $X$ in the space $\mathbb{R}^n$ and if $n = 2$ then $X^\circ$ is the interior of $X$ in the plane $\mathbb{R}^2$. For abbreviation let $I = [0,1]$ and $J = ]0,1[$ and let $2\mathbb{N} := \{2k \mid k \in \mathbb{N}\}$ be the set of all positive even numbers. Furthermore, if $X \subset \mathbb{R}^2$ and $L \subset \mathbb{R}$ we regard $X \times L^k$ as a subset of $\mathbb{R}^{k+2}$ for every $k \geq 0$ where $X \times L^k$ is identified with $X$ if $k = 0$.

For each $m \in \mathbb{N}$ let $D_m := [2^{-m}, 2^{-m+1}]^2$ and

$$W_m := \bigcup_{k=1}^{m} \left[ 2^{-m} + \frac{2k-1}{2m+1} \cdot 2^{-m}, \quad 2^{-m} + \frac{2k}{2m+1} \cdot 2^{-m} \right]^2.$$

So $D_m$ is a compact square area and $W_m$ is a union of $m$ disjoint compact square areas which all lie in the interior of $D_m$. For $S \subset 2\mathbb{N}$ put

$$Z_S := [-1,0]^2 \,\cup\, \bigcup_{m \in S} (D_m \setminus W_m^\circ)$$

and define $\mathcal{F}_n := \{Z_S^\circ \times J^{n-2} \mid S \subset 2\mathbb{N}\}$ for each dimension $n \geq 2$. Clearly, we always have $\overline{Z_S^\circ} = Z_S$ and hence $\overline{Z_S^\circ \times J^{n-2}} = Z_S \times I^{n-2}$. Obviously, for every $U \in \mathcal{F}_n$ the closure $\overline{U}$ is compact and a union of cubes $[a_1, a_1 + h] \times \cdots \times [a_n, a_n + h]$ with $h > 0$ so that the interiors of distinct cubes are always disjoint. (Note that

$(D_m \setminus W_m^\circ) \times I^{n-2}$ is a union of precisely $((2m+1)^2 - m)(\frac{1}{l})^{n-2}$ such cubes with edge length $l = \frac{2^{-m}}{2m+1}$.)

In order to verify that the family $\mathcal{F}_n$ has the desired homeomorphism properties it is enough to investigate the components of the space $Z_S^\circ \times J^{n-2}$ and $Z_S \times I^{n-2}$ respectively. Clearly the components are always path connected spaces and so it is natural to determine their fundamental groups. (Two spaces $X, Y$ cannot be homeomorphic if the fundamental group of some path component of $X$ is not isomorphic to the fundamental group of any path component of $Y$.)

For each $S \subset 2\mathbb{N}$ the components of the space $Z_S \times I^{n-2}$ resp. $Z_S^\circ \times J^{n-2}$ are precisely $(D_m \setminus W_m^\circ) \times I^{n-2}$ resp. $(D_m^\circ \setminus W_m) \times J^{n-2}$ with $m \in S$ and the one simply connected component $[-1,0]^2 \times I^{n-2}$ resp. $]-1,0[^2 \times J^{n-2}$.

For each $m \in \mathbb{N}$ the fundamental group both of $D_m \setminus W_m^\circ$ and of $D_m^\circ \setminus W_m$ is free on $m$ generators. This is enough since for $n \geq 3$ both $I^{n-2}$ and $J^{n-2}$ have trivial fundamental groups. (If $X, Y$ are path connected spaces then the fundamental group of the product space $X \times Y$ is isomorphic to the direct product of the fundamental groups of $X$ and $Y$.)

Finally, dispensing with compactness, it is plain to modify the definition of $\mathcal{F}_n$ so that each member of $\mathcal{F}_n$ is the interior of a union of cubes of the form $[k_1, k_1 + 1] \times \cdots \times [k_n, k_n + 1]$ with $k_1, \ldots, k_n \in \mathbb{Z}$. For example, for $\emptyset \neq S \subset 2\mathbb{N}$ replace $Z_S^\circ \times J^{n-2}$ with $Y_S^\circ \times J^{n-2}$ where $Y_S := \bigcup_{m \in S} \{(t_m x, t_m y) \mid (x,y) \in D_m \setminus W_m^\circ\}$ with $t_m := 4^m(2m+1)$.

## 6. Proofs of Propositions 1 and 3

First we need two basic lemmas.

**Lemma 1.** *If $g$ is an arbitrary injection from $\mathbb{N}$ into $\mathbb{N}$ then $\{n \in \mathbb{N} \mid n \leq g(n)\}$ must be an infinite set.*

*Proof.* Suppose that $\{n \in \mathbb{N} \mid n \leq g(n)\}$ is a finite set. Then we may fix $N \in \mathbb{N}$ so that $n > g(n)$ for every $n \geq N$. Let $M := \max\{g(n) \mid n = 1, 2, \ldots, N\}$. Then $M \geq N$ since $g$ is injective. Consequently, $g(M+1) \leq M$ and $g(n) < n \leq M$ whenever $N \leq n \leq M$ and $g(n) \leq M$ whenever $n \leq N$. But then $g$ maps the set $\{1, 2, \ldots, M, M+1\}$ into the set $\{1, 2, \ldots, M\}$ and hence $g$ cannot be injective, *q.e.d.*

**Lemma 2.** *If $a, b \in \mathbb{R}$ and $a < b$ then for any family $\mathcal{F}$ of mutually disjoint intervals $[x,y]$ with $x < y$ the equality $[a,b] = \bigcup \mathcal{F}$ is only possible in the trivial case $\mathcal{F} = \{[a,b]\}$.*

Lemma 2 is a consequence of a well-known theorem due to Sierpiński (cf. [1], 6.1.27). A direct proof of Lemma 2 is a nice exercise. (For instance, $[a,b] =$

$\bigcup \mathcal{F}$ implies that the set $P$ of all boundary points $p \notin \{a,b\}$ of intervals in $\mathcal{F}$ is countable and perfect, whence $P = \emptyset$.)

In order to prove Proposition 1 it is enough to settle the statement on the closed point sets $X_u$ because each $X_u$ is a completion of the metric space $X_u^\circ$. Fix $2 \leq v < w$ and let $I_n := \left[2^{v^n}, 2^{v^n} + v^n\right] =: [a_n, b_n]$ and $J_n := \left[2^{w^n}, 2^{w^n} + w^n\right] =: [c_n, d_n]$ for every $n \in \mathbb{N}$. So we have $b_n - a_n = v^n$ and $d_n - c_n = w^n$ for every $n \in \mathbb{N}$ and $1 + b_n \leq a_{n+1}$ and $1 + d_n \leq c_{n+1}$ for every $n \in \mathbb{N}$. Assume indirectly that $f$ is a uniformly continuous bijection from $X_v$ onto $X_w$. We claim that for each $n \in \mathbb{N}$ we must have $f(I_n) = J_m$ for some $m \in \mathbb{N}$. Indeed, choose $m$ so that $f(I_n) \cap J_m \neq \emptyset$ and define an equivalence relation on $J_m$ via $x \sim y$ if and only if $f^{-1}(x), f^{-1}(y) \in I_k$ for some $k$. Then, since $f(I_k)$ is always compact and connected and since all point sets $J_k$ are open and closed in the space $X_w$, the family $\mathcal{F}$ of all equivalence classes must equal $\{f(I_k) \mid k \in K\}$ for some $K \subset \mathbb{N}$ with $n \in K$. Hence, in view of Lemma 2 we must have $K = \{n\}$ or, equivalently, $f(I_n) = J_m$.

Consequently, there is a bijection $g\colon \mathbb{N} \to \mathbb{N}$ such that $f(I_n) = J_{g(n)}$ for every $n \in \mathbb{N}$. By Lemma 1, $G := \{n \in \mathbb{N} \mid n \leq g(n)\}$ is an infinite set. Let $Y_v := \bigcup_{n \in G} I_n$ and $Y_w := \bigcup_{n \in G} J_{g(n)}$. Then $f$ is a uniformly continuous function from the unbounded set $Y_v$ onto the unbounded set $Y_w$. Thus we may fix $0 < \delta < 1$ so that $|f(x) - f(y)| \leq 1$ whenever $x, y \in Y_v$ and $|x - y| \leq \delta$. Naturally, $f$ is strictly monotonic on each interval $I_n (n \in G)$ and $\{f(a_n), f(b_n)\} = \{c_{g(n)}, d_{g(n)}\}$ for every $n \in G$. Now, for every $n \in G$ we have

$$\begin{aligned} w^n \leq w^{g(n)} &= d_{g(n)} - c_{g(n)} = |f(a_n) - f(b_n)| \\ &= |f(a_n) - f(a_n + \delta)| + |f(a_n + \delta) - f(a_n + 2\delta)| + \cdots + |f(a_n + k\delta) - f(b_n)| \\ &\leq k + 1 \end{aligned}$$

where $k \in \mathbb{N}$ is chosen so that $a_n + k\delta < b_n \leq a_n + (k+1)\delta$ or, equivalently, $k\delta < v^n \leq (k+1)\delta$. But then $w^n - 1 \leq v^n/\delta$ for every $n$ in the infinite set $G$. This is impossible since $\lim_{n \to \infty} w^n/v^n = \infty$ and so the proof of Proposition 1 is finished.

*Remark.* Concerning higher dimensions, in view of the preceding proof it is plain that the $c$ closed resp. open point sets $\bigcup_{m=1}^{\infty} [2^{u^m}, 2^{u^m} + u^m]^n$ resp. $\bigcup_{m=1}^{\infty} ]2^{u^m}, 2^{u^m} + u^m[^n$ $(u \geq 2)$ in $\mathbb{R}^n$ are metrically distinct and topologically similar for arbitrary $n$.

Now we are going to prove Proposition 3. As usual, the distance $d(A,B)$ between two nonempty subsets $A, B$ of $\mathbb{R}^n$ is the infimum of all numbers $d(a,b)$ with arbitrary $a \in A$ and $b \in B$ where $d(x,y)$ denotes the Euclidean distance between $x, y \in \mathbb{R}^n$.

If $U$ is an open subspace of $\mathbb{R}^n$ and if $\mathcal{G}$ is the (countable) family of all components of $U$, then let us call a finite subset $\mathcal{F}$ of $\mathcal{G}$ a *chain* if and only if there is an ordering $\mathcal{F} = \{U_1, \ldots, U_m\}$ with $U_i \neq U_j$ $(i \neq j)$ and $d(U_k, U_{k+1}) = 0$ for every

$k < m$. The *length* of $\mathcal{F}$ is $m$. A chain is *maximal* if it is not contained in a chain of greater length. For every $S \subset \mathbb{N}$ all the components of $Y_S$ are open cubes of the form $]a_1, a_1 + h[\times \cdots \times ]a_n, a_n + h[$ and, evidently (by induction on the dimension $n$),

$$Y_S = \bigcup_{s \in S} \big(\bigcup \mathcal{F}_s\big) \,\cup\, \bigcup_{k=1}^{\alpha} C_k \qquad (\alpha \in \mathbb{N} \cup \{\infty\})$$

where $\{C_k\}$ is always a maximal chain of length 1 and $\mathcal{F}_s$ is a maximal chain of length $(s+1)^n$ for every $s \in S$ and $d(\bigcup \mathcal{F}_i, \bigcup \mathcal{F}_j) > 0$ whenever $i, j \in S$ and $i \neq j$.

Suppose that $S, S' \subset \mathbb{N}$ and that $f$ is a uniform homeomorphism from $Y_S$ onto $Y_{S'}$. Of course, $f(C)$ is a component of $Y_{S'}$ if and only if $C$ is a component of $Y_S$. Furthermore, for any $\emptyset \neq A, B \subset Y_S$ we certainly have $d(A,B) = 0$ if and only if $d(f(A), f(B)) = 0$. Therefore, for every $s \in S$ the set $\{f(C) \mid C \in \mathcal{F}_s\}$ must be a maximal chain of length $(s+1)^n$ in the space $Y_{S'}$, whence $s \in S'$. Thus $S \subset S'$. Similarly, $S' \subset S$.

## 7. Proof of Proposition 2

For a nonempty set $X$ in the family $\mathcal{U} \cup \mathcal{A}$ let $\mathcal{C}(X)$ be the family of all components of the Euclidean subspace $X$ of $\mathbb{R}$. Since each member of $\mathcal{C}(X)$ is an open or closed interval, we may define a natural strict linear ordering $\prec$ of $\mathcal{C}(X)$ via $A \prec B$ for distinct (and hence disjoint) $A, B \in \mathcal{C}(Z)$ if and only if $a < b$ for *some* $(a,b) \in A \times B$ or, equivalently, if $a < b$ for *every* $(a,b) \in A \times B$.

Let $\emptyset \neq X, Y \subset \mathbb{R}$ and let $\varphi\colon X \to Y$ be an order isomorphism. Then $\varphi$ is a homeomorphism with respect to the order topologies of $(X, <)$ and $(Y, <)$. Moreover, if the sets $X, Y$ lie in the family $\mathcal{U} \cup \mathcal{A}$ then $\varphi$ is a homeomorphism between the Euclidean spaces $X$ and $Y$ and hence $A \mapsto \varphi(A)$ defines a bijection from $\mathcal{C}(X)$ onto $\mathcal{C}(Y)$ and it is evident that this bijection is an order isomorphism between $(\mathcal{C}(X), \prec)$ and $(\mathcal{C}(Y), \prec)$. Thus, $X, Y \in \mathcal{U} \cup \mathcal{A}$ are not order-isomorphic if the two families $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are not order-isomorphic. (Conversely, if either $X, Y \in \mathcal{U}$ or $X, Y \in \mathcal{A}$ are compact then from any order isomorphism between $(\mathcal{C}(X), \prec)$ and $(\mathcal{C}(Y), \prec)$ we may easily construct an order isomorphism between $(X, <)$ and $(Y, <)$. This is not true for arbitrary sets $X, Y \in \mathcal{A}$ or for compact sets $X, Y \subset \mathbb{R}$. Consider, for example, $X = [0,1] \cup [2,3]$ and firstly $Y = [0,1] \cup [2,\infty[$ and secondly $Y = [0,1] \cup \{2\}$.)

So in order to settle Proposition 2 it is enough to find $c$ metrically similar sets $X$ in the family $\mathcal{U}$ resp. $\mathcal{A}$, such that the corresponding sets $\mathcal{C}(X)$ are mutually not order-isomorphic. Let $\mathcal{G}$ be the family of all functions $g$ from $\mathbb{N}$ to $\{0,1\}$ such that the set $g^{-1}(\{1\})$ is infinite. Clearly, the cardinal number of the family $\mathcal{G}$ is $c$.

For every $n \in \mathbb{N}$ define

$$Z_n := \bigcup_{k=1}^{\infty} \Big( \, ]6n+1+3^{-2k}, 6n+1+3^{-2k+1}[ \, \cup \, ]6n+2-3^{-2k+1}, 6n+2-3^{-2k}[ \, \Big)$$

and for every $g \in \mathcal{G}$ define

$$U_g := \bigcup_{n=1}^{\infty} \Big( Z_n \cup \, ]6n, 6n+1[ \, \cup \, ]6n+2, 6n+3[ \, \Big) \cup \bigcup_{n \in g^{-1}(\{1\})} ]6n+4, 6n+5[.$$

It is plain that the $c$ open sets $U_g (g \in \mathcal{G})$ are metrically similar and that the $c$ closed sets $\overline{U_g} (g \in \mathcal{G})$ lie in the family $\mathcal{A}$ and are metrically similar too. Let $\zeta$ denote the order type of $\mathbb{Z}$. Then $\zeta$ is also the order type of $\mathcal{C}(Z_n)$ for every $n \in \mathbb{N}$ and it is evident that for each $g \in \mathcal{G}$ both the order type of $\mathcal{C}(U_g)$ and the order type of $\mathcal{C}(\overline{U_g})$ equals

$$\begin{aligned} &(1+\zeta+1)+g(1)+(1+\zeta+1)+g(2)+(1+\zeta+1)+g(3)+\cdots \\ = \, &1+\zeta+(1+g(1)+1)+\zeta+(1+g(2)+1)+\zeta+(1+g(3)+1)+\cdots \end{aligned}$$

where a nonnegative integer $k$ is always the order type of any linearly ordered set of precisely $k$ elements. (If $\alpha, \beta$ are order types of nonempty sets then $\alpha+0+\beta$ is just $\alpha+\beta$.) Naturally, for distinct $f, g \in \mathcal{G}$ the order types of $\mathcal{C}(U_f)$ and $\mathcal{C}(U_g)$ are distinct and this concludes the proof.

*Remark.* In view of the previous considerations it is easy to track down $c$ metrically similar compact subsets of $\mathbb{R}$ without singleton components which are mutually not order-isomorphic. (The existence of such sets follows from [5] Main Theorem 2.) Take for example (with $h$ as in the proof of Theorem 1) the $c$ point sets $h(\overline{U_g}) \cup [1,2]$ $(g \in \mathcal{G})$.

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# Alexander Grothendieck 1928–2014

**Johannes Wallner**
TU Graz

Am 13. November 2014 starb Alexander Grothendieck in Saint-Girons (Ariège) nahe dem Dorf Lasserre, wo er seit den 1990er-Jahren zurückgezogen lebte. Er wird von manchen als der größte „reine“ Mathematiker des 20. Jahrhunderts bezeichnet, aber in diesem Artikel soll gar nicht versucht werden, sein bahnbrechendes mathematisches Werk in der algebraischen Geometrie und anderen Gebieten im Detail zu würdigen. Der Autor fühlt sich dazu nicht berufen und verweist lieber auf den für *Nature* intendierten Nachruf von David Mumford und John Tate in seiner ursprünglichen Fassung [8]. Stattdessen soll ein Lebenslauf geschildert werden, der von Genius, Kompromisslosigkeit und, in späteren Jahren, von psychischen Instabilitäten gezeichnet ist. Alle drei Aspekte sind in Grothendiecks umfangreichen Schriften ausführlich dokumentiert.

Seine Eltern und die Umstände seiner Jugend waren zweifelsohne prägend für Grothendieck. Der Vater, Sascha Shapiro, war ein russischer Photograph und militanter Anarchist, der in Deutschland die Journalistin Hanka Grothendieck kennenlernte. Ihr 1928 geborener Sohn Alexander verbrachte die Jahre 1934–1939 in Hamburg, während seine Eltern im Spanischen Bürgerkrieg aktiv waren. Er sah sie erst 1939 in Frankreich wieder. Die Zeit nach Ausbruch des Zweiten Weltkriegs war nicht leicht für die Familie, die zuerst als Deutsche und dann, im Vichy-Regime, als Juden unerwünscht waren. Alexanders Vater Shapiro kam 1942 in Auschwitz um. Dem jungen Alexander Grothendieck gelang es jedoch, eine Schule zu besuchen und als 16-Jähriger an der Universität Montpellier die Zulassung zum Studium zu erreichen. Die Prüfungsarbeit war unleserlich, die zuständigen Professoren legten dem jungen Genius jedoch keine Steine in den Weg [7]. Er war ein brillanter Student und dissertierte schließlich bei Dieudonné.

Eine berufliche Laufbahn von Alexander Grothendieck in Frankreich wurde durch seine Staatenlosigkeit behindert, und als Wehrdienstverweigerer wollte er auch nicht um die französische Staatsbürgerschaft ansuchen. So lehrte er ab 1953 in Brasilien und in den USA. Ein großer Glücksfall war die Stiftung des *Institut des hautes études scientifiques* (IHES) in Bures-sur-Yvette durch den Industriellen und Mathematiker Léon Motchane, der von der Persönlichkeit des damals 27-jährigen Alexander Grothendieck beeindruckt war. 1959 begann für diesen am

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 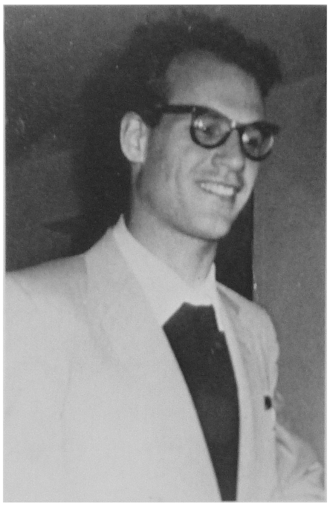

*Links:* Grothendieck 1940 im Internierungslager von Rieucros. *Rechts:* 1955 in den USA, photographiert von Paul Halmos (© American Math. Society. Mit freundlicher Genehmigung).

IHES eine sehr produktive Schaffensperiode, die bis 1970 anhalten sollte. Seine grundlegend neue Sicht der algebraischen Geometrie ist in dem umfangreichen Bänden *Éléments de géométrie algébrique* [11] und *Séminaire de Géométrie Algébrique du Bois Marie* [12, 13, 14, 15, 16, 17, 18, 19, 20] publiziert.

Alexander Grothendieck gab zeit seines Lebens seinen politischen Überzeugungen Ausdruck. Während des Vietnamkriegs hielt er in den Wäldern rund um Hanoi Vorlesungen über Kategorietheorie, um gegen den amerikanischen Imperialismus zu protestieren. Er besuchte nur Tagungen, die keine Unterstützung von militärischen Institutionen erhielten. Im Jahr 1966 erhielt er die Fields-Medaille, lehnte es jedoch ab, nach Moskau zu reisen, um sie ihn Empfang zu nehmen. Auf diese Weise wollte er gegen die Verfolgung von Dissidenten in der Sowjetunion protestieren. In den 1970ern bezeichnete er sich als „militanter Aktivist" und gründete gemeinsam mit zwei Mathematikerkollegen, Claude Chevalley und Pierre Samuel, die Gruppe *Survivre et Vivre*, eine Antikriegs-, Antiimperialismus- und Umweltbewegung.

Er schreckte auch nicht davor zurück, für seine Überzeugungen die eigene berufliche Existenz aufs Spiel zu setzen: 1970 entdeckte er, dass das IHES, wenn auch nur in geringem Ausmaß und auf indirekte Weise, vom französischen Verteidigungsministerium finanziert wurde. Als es ihm nicht gelang, diesen Geldfluss zu unterbinden, trat er aus Protest von seiner Position zurück. Möglicherweise hat auch die Leitung des IHES diesen Konflikt ausgenützt, um einen mittlerweile unbequemen Kollegen loszuwerden [10]. Jedenfalls gelang es ihm in weiterer Folge nicht, eine vergleichbare Position oder zumindest eine Position an einer Spitzenuniversität zu erhalten – entweder weil seine Forderungen zu hoch waren oder weil potentielle Arbeitgeber bereits argwöhnisch und zurückhaltend wurden. Er fasste in der akademischen Welt nicht mehr richtig Fuß bzw. wollte dies vielleicht auch gar nicht mehr. Das Collège de France bot ihm eine temporäre Position an, die

Alexander Grothendieck am IHES (Photo mit freundlicher Genehmigung des IHES).

er als politische Bühne nutzte. 1973 wurde er Professor an der Universität Montpellier, wo er hauptsächlich Vorlesungen für Studienanfänger hielt. Er kehrte in den Jahren 1984 bis 1988 auch ans CNRS zurück. Im Jahr 1988 erhielt er den mit $160.000,– dotierten Crafoord-Preis der Schwedischen Akademie der Wissenschaften; in einem Leserbrief an *Le Monde* [6] legte er dar, dass er ihn aufgrund der Unehrlichkeit und Korruption des wissenschaftlichen und politischen Establishments nicht annehmen könne.

Seit den 1980er-Jahren waren Alexander Grothendiecks Schriften mehrheitlich nicht mehr mathematischen Inhalts, wie zum Beispiel die autobiographischen Notizen [4] („ein Gemisch aus philosophischen Beleidigungen, paranoiden Attacken auf Kollegen und hie und da auch genialen Einsichten" [1]). In dem Manuskript [5] schreibt er, wie er durch die Entdeckung der Bedeutung von Träumen von der Existenz Gottes überzeugt wurde. Für mehr Informationen sei auf *http://www.grothendieckcircle.org* verwiesen, wo man eine Fülle an Material und Schriften von und über Alexander Grothendieck finden kann. Einen sehr ausführlichen und in die Tiefe gehenden Bericht über Leben, Schriften, und persönliche Ziele gibt auch W. Scharlaus *Oberwolfach Lecture* 2006 [10].

Grothendieck war ab den 1980ern psychisch nicht mehr stabil. 1991 verbrannte er Tausende Manuskriptseiten und zog sich an einen unbekannten Ort (d.h., Lasserre) zurück. Er hatte keinen Kontakt mehr mit seiner Familie und mit Kollegen. Ab diesem Zeitpunkt kursierten verschiedene Gerüchte um ihn. Man machte sich auch auf, ihn zu suchen. Leila Schneps, Koautorin (gemeinsam mit Winfried Scharlau) einer noch unvollständigen Biographie, fand ihn „besessen vom Teufel, den er überall in der Welt am Werk sieht“. Ein Lebenszeichen von Alexander Grothendieck gab es 2010, als er in Briefen verlangte, dass alle seine Schriften aus Bibliotheken entfernt werden. Er lebte an seinem Rückzugsort bis kurz vor seinem Tod am 13.11.2014. Es erschienen Nachrufe im *Telegraph* [1], in *Le Monde* [2], der *New York Times* [3] und in *Nature* [9, 8].

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# Buchbesprechungen

**E. Casas-Alvero: Analytic Projective Geometry.** (EMS Textbooks in Mathematics) EMS, Zürich, 2014, xvi+620 S. ISBN 978-3-03719-138-5 H/b € 58,–.

Most current textbooks on projective geometry are rather limited in scope. Often, the focus on planar geometry only or they omit considerable portions of classical results that might still be useful in nowadays's research. This book by Eduardo Casas-Alvero does not suffer from these shortcomings. It is truly comprehensive. Assuming just a basic knowledge in linear algebra, the author develops virtually all aspects of classical real and complex projective geometry and its relation to affine and metric geometry. This makes this book a valuable reference for classical projective geometry, probably unsurpassed in this respect by any other textbook published within the last 50 years. Casas-Alvero presents the usual topics like cross ratio, projective transformations, classification results for quadrics, or aspects of non-Euclidean geometry in Cayley-Klein models but also classical topics that cannot easily be found elsewhere in modern literature, among them the geometry of the projective space of quadrics, classification of collineations, or projective geometry for artists.

The style is meticulous and precise. The book can well be used as a basis for a course on projective geometry but then a careful selection of material and proofs is advisable. Sometimes one can resort to more direct techniques from linear algebra to abbreviate proofs or other derivations. The author often avoids them in favor of purely projective or coordinate-free methods.

At the end of each chapter we find numerous exercises that, without being overly difficult, often introduce the reader to interesting and particularly nice results. Unfortunately, their proofreading was less thorough than for the rest of the text ("tetrahedra of $\mathbb{P}_3$, with vertices $p_1, \ldots, p_6$"). This should not be a problem for the learned reader but might confuse beginning students. The same is true for the author's occasional use of nonstandard terminology. In order to find out what "equinharmonic" points are, one probably has to pick up this book – but this is highly recommend to anybody using projective geometry in research.

H.-P. Schröcker (Innsbruck)

**L. J. Halbeisen: Combinatorial Set Theory.** With a Gentle Introduction to Forcing. (Springer Monographs in Mathematics.) Springer, Berlin, Heidelberg, 2012, xvi+453 S. ISBN 978-1-4471-2172-5 H/b € 89,95.

Wer hier ein eher traditionelles Lehrbuch über Kombinatorik erwartet, wird verwundert sein, dass sich praktisch der gesamte Text mit „unendlicher Kombinatorik“ beschäftigt. Hauptthemen sind die Zermelo-Fränkelsche Mengenlehre (ZF), die ZF mit Zermelos Auswahlaxiom (ZFC) und kombinatorische Aussagen im Rahmen dieser Theorien. Als „Overture“ beginnt das Buch mit dem (unendlichen) Satz von Ramsey: Wenn man, für eine fixe natürliche Zahl $n$, alle Teilmengen der natürlichen Zahlen mit $n$ Elementen mit endlich vielen Farben färbt, dann gibt es eine unendliche Menge $M$ natürlicher Zahlen, sodass alle $n$-elementigen Teilmengen in $M$ die selbe Farbe besitzen. Dies ist eine unendliche Version des Schubfachprinzips.

Sodann werden die Theorien ZF und ZFC eingehend untersucht. Besonders genau wird die Technik des Forcing vorgestellt. Im Prinzip geht es dabei darum, die Konsistenz eines Satzes mit ZFC so zu beweisen, dass man ein Modell der ZFC so adaptiert, dass dieser Satz auch in dem adaptierten Modell gilt. Wenn auch die Negation des Satzes konsistent ist, so ist der Satz als unabhängig von ZFC bewiesen. Dies wird z.B. mit „Martin's Axiom" durchgeführt. Viele weitere Sätze behandeln die Gleichheit oder Ungleichheit von Kardinalzahlen in verschiedenen Modellen der Mengenlehre.

Der Text ist sehr angenehm geschrieben und enthält sehr ausführliche historische Hintergrundinformationen. Wenn eine Leserin oder ein Leser gute Grundkenntnisse der endlichen Kombinatorik hat, so ist dies gewiss kein Fehler und man kann die Aussagen dieses Buchs wesentlich besser einordnen und schätzen.

G. Pilz (Linz)

**B. C. Hall: Quantum Theory for Mathematicians.** (Graduate Texts in Mathematics) Springer New York, Heidelberg, Dordrecht, London 2013, xvi+554 S. ISBN 978-1-4614-7115-8 H/b € 59,49.

This textbook is meant for advanced studies on quantum mechanics for a mathematical readership. The exercises at the end of each chapter make the book especially valuable.

The first three chapters of this monograph introduce to quantum mechanics. Chapter 4 and 5 deal with the free Schrödinger equation and a particle in a square well. Chapters 6–10 deal with the spectral theorem, chapter 11 deals with the harmonic oscillator. The uncertainty principle, quantization schemes, and the Stone-von Neumann theorem are the topics of the following three chapters. Chapter 15 is devoted to so-called WKB approximation, Chapter 16 to Lie groups and Lie algebras. In Chapter 17 angular momentum and spin are studied. Chapter 18 describes energy levels of hydrogen atoms and Chapter 19 composite systems. Chapter 20 develops the path integral formulation of quantum mechanics. Chapter 21 is entitled Hamiltonian mechanics on manifolds. Chapters 22 and 23 consider geometric quantization.

A. Winterhof (Linz)

**K. H. Hofmann, S. A. Morris: The Structure of Compact Groups. A Primer for the student – a handbook for the expert.** (Studies in Mathematics 25.) De Gruyter, Berlin, 2013, xxii+924 S. ISBN 978-3-11-029655-6 H/b € 149,95.

This is a monumental work with more than 900 pages on the theory of compact groups and their applications. It is already the third edition (the previous ones appeared in 1998 and 2006) and it brings the reader to the present state of the art.

The book is an extensive source of information for graduate students and researchers and is to a great extent self-contained and can also serve as a handbook on this huge subject. It starts from the foundations of topological groups, moves on to representations of compact groups and their characters, gives a detailed account on the contributions of Peter and Weyl and then concentrates on (compact) Lie groups and compact abelian groups, since the approximation of arbitrary compact groups by Lie groups (via projective limits) is a big issue (which, however, is even improved in this text).

The structure of compact groups and compact group actions is then displayed in detail and mastership. To the existing appendices on abelian and topological groups and categories, two more (measures on compact groups and projective limits) were added. Important theorems are clearly highlighted. Many exercises (often with hints) are inserted when they fit best; each chapter ends with references and hints for additional reading. A very useful and detailed index also helps a lot. Altogether, this book is a must for everybody who is interested in topological groups.

G. Pilz (Linz)

**J. Neukirch: Class Field Theory.** (The Bonn Lectures ed. by A. Schmidt.) Springer, Berlin, New York, Dordrecht, London, 2013 x+184 S. ISBN 978-3-642-35436-6 P/b € 38,45.

This book is a translation by F. Lemmermeyer and W. Snyder of the German original from 1967.

It is a very nice introduction to class field theory and the reader is only assumed to be familiar with the basics of algebraic number theory. The approach is based on cohomology.

The three chapters of the book contain an introduction to group cohomology, local and global class field theory.

A. Winterhof (Linz)

**S. Schiemann, R. Wöstenfeld: Die Mathe-Wichtel Band 1. Humorvolle Aufgaben mit Lösungen für mathematisches Entdecken ab der Grundschule – Band 2. Humorvolle Aufgaben mit Lösungen für mathematisches Entdecken ab der Sekundarstufe.** (Sachbuch) Springer Spektrum, 2014, 131 S. ISBN 978-3-658-03072-8 P/b € 18,49 – 179 S. ISBN 978-3-658-03074-2 P/b € 17,99.

Die beiden Bände präsentieren eine Sammlung mathematischer Fragestellungen, die in altersgerechte, interessante „Wichtelgeschichten" verpackt sind. Damit sollen Schülerinnen und Schüler spielerisch und humorvoll an mathematische Probleme und deren Lösungen herangeführt werden. Im ersten Band sind Aufgabenstellungen enthalten, die ab dem Grundschulniveau Lust auf Mathematik wecken sollen, während der zweite Band Aufgaben umfasst, die ab der Sekundarstufe zugänglich sein sollten. Die Aufgaben sind oft humorvoll in Geschichten eingekleidet. Damit scheint das Ziel durchaus erreichbar, mit diesen Aufgaben bei Schülerinnen und Schülern schon früh das Interesse an Mathematik zu wecken. Die Aufgaben enthalten zumeist einen Einleitungsteil mit einer Wichtelgeschichte, dann die mathematische Fragestellung und zuletzt vier mögliche Antworten, aus denen die Schülerinnen und Schüler auswählen können. Ihrem Einsatz im Schulunterricht oder in manchem Projekt steht meiner Meinung nach nichts im Wege. In beiden Bänden finden sich im hinteren Teil jeweils Lösungsvorschläge für alle Aufgaben sowie „Anregungen zum Weiterdenken". Die beiden Bücher können damit allen empfohlen werden, die Ideen für ihren Mathematikunterricht oder für die Erstellung von interessanten und altersgerechten mathematischen Aufgabenstellungen suchen.

O. Röschel (Graz)

**D. Schleicher, M. Lackmann (Hg.): Eine Einladung in die Mathematik.** Einblicke in aktuelle Forschung. Aus dem Engl. übersetzt von B. Arnold, R. Stoll und M. Oliver. Springer Spektrum Berlin Heidelberg 2013, xv+228 S.,ISBN 978-3-642-25797-1 P/b € 25,65.

Das Buch ist während der 50. IMO 2009 entstanden. Einer der Herausgeber war Hauptorganisator, der andere ein Teilnehmer. Es enthält 15 von einander unabhängige Beiträge. Zwei davon (Ramseytheorie, Spiele) beleuchten nebenbei auch Gemeinsamkeiten und Unterschiede zwischen IMO-Aufgaben und mathematischer Forschung. Die übrigen Kapitel behandeln folgende Themen: Primzahlen, Diophantische Gleichungen, Quadratische Formen, Dynamische Systeme, Graphentheorie, Komplexität von Kommunikation, Numerische Problemstellungen, Strömungsmechanik, Hardy-Ungleichung, Verfolgungskurven, Mathematische Wettbewerbe (Berechnung von Pi, Schach, Tetraederpackungen), Komplexe Dynamik und Mandelbrotmengen.

Die Kapitel führen direkt an jüngste Forschungsergebnisse heran. Beweise sind zwangsläufig eher Mangelware, einige Autoren versuchen aber zumindest plausible Erklärungen anzubieten.

Erstsemestrigen, die mit einer mathematisch-wissenschaftlichen Laufbahn liebäugeln, kann man dieses Buch als ersten Einblick in unterschiedlichste Richtungen und Methoden mathematischer Forschung sehr empfehlen.

M. Kronfellner (Wien)

**M. Trifković: Algebraic Theory of Quadratic Numbers.** (Universitext) Springer, New York, Heidelberg, Dordrecht, London, 2013 xi+197 S. ISBN 978-1-4614-7716-7 P/b € 54,99.

Wie der Titel des Buchs schon sagt, werden in hier die algebraischen Aspekte der quadratischen Zahlkörper behandelt. Hauptziel des Buchs ist, die elementare algebraische Zahlentheorie für quadratische Zahlkörper zu entwickeln. So werden die folgenden Themen behandelt: Gitter, eindeutige Primidealfaktorisierung, Verzweigungstheorie, Idealklassengruppen, Kettenbrüche, Einheitengruppe und quadratische Formen (alles beschränkt auf die quadratischen Zahlkörper).

Der Autor kommt mit minimalen Voraussetzungen der Algebra aus, um die oben genannten Themen zu behandeln. Mit vielen Beispielen und ausführlichen Beweisen wird dem Leser die Thematik behutsam näher gebracht. Dabei helfen auch zahlreiche Illustrationen und eine Reihe von Übungsbeispielen am Ende jedes Abschnitts.

Das Buch kann man jedem empfehlen, der schon erste Kontakte mit Algebra und Zahlentheorie hatte und in die algebraische Zahlentheorie hineinschnuppern möchte. Es eignet sich zum Selbststudium für Studenten oder auch als Grundlage eines Seminars oder einer Vorlesung zu diesem Thema.

V. Ziegler (Salzburg)

# Nachrichten der Österreichischen Mathematischen Gesellschaft

**Protokoll der Generalversammlung der ÖMG am 21.11.2014, Univ. Wien**

Freitag, 21.11.2014, 17:15–18:15 Uhr (Hörsaal 13 der Fakultät für Mathematik)

*Tagesordnung:*

1. Begrüßung und Feststellung der Beschlussfähigkeit
2. Berichte des Vorsitzenden, des Kassiers und weiterer Vorstandsmitglieder
3. Bericht der Rechnungsprüfer und gegebenenfalls Entlastung des Vorstands
4. Berichte aus den Landessektionen und den Kommissionen
5. Wahl des Beirats und der Vorsitzenden der Landessektionen
6. Verleihung der Studienpreise und des Förderungspreises
7. Allfälliges

**1. Begrüßung und Feststellung der Beschlussfähigkeit.** Der Vorsitzende begrüßt die Anwesenden. Die Beschlussfähigkeit wird festgestellt.

**2. Berichte des Vorsitzenden, des Kassiers und weiterer Vorstandsmitglieder**.

*Mitgliederstand:* Derzeit hat die ÖMG 595 persönliche Mitglieder, davon 10 nicht zahlende Mitglieder (6 Ehrenmitglieder, 4 Mitglieder sind wegen Bedürftigkeit befreit) und 19 institutionelle Mitglieder. Es gab seit 24.9.2013 22 Beitritte und 21 Austritte, inklusive einem Zusendestopp für 9 Mitglieder wegen mehrmaligen Nichtzahlens des Mitgliedsbeitrags (entspricht einem Austritt).

In einer Schweigeminute gedenken die Anwesenden der sechs der ÖMG bekannten Todesfälle: *Heinrich Lakatha*, gest. 20.3.2013, *Wolfgang Schwarz*, gest. 19.7.2013, *Hans Sachs*, gest. 15.12.2013, *Wilfried Hazod*, gest. 23.5.2014, *Risak Veith*, gest. 10.7.2014 und *Werner Mark*, gest. 6.9.2014.

*Tagungen:* Die nächste Tagung der ÖMG, gemeinsam mit der János Bolyai-Mathematischen Gesellschaft, findet von 25.–27.8.2015 in Győr (Ungarn) statt. Es wurde ein *steering committee* gebildet: Michael Drmota, Michael Oberguggenberger (Vorsitzender der ÖMG), Zoltán Horváth (als Vertreter des Veranstaltungsorts, der Univ. Győr), András Recski (Schriftführer, Bolyai-Ges.), Gyula Katona (Vorsitzender der Bolyai-Ges.).

Die Mitglieder des Programmkomitees sind Henk Bruins, Mihyun Kang, Robert Tichy und Alexander Ostermann aus Österreich sowie László Gerencsér, Ervin Győri, Attila Pethő und Domokos Szász aus Ungarn. Eingeladene Vortragende sind die folgenden:

– Béla Bollobás (Diskrete Mathematik, Univ. Cambridge)
– Ulrich Langer (Numerische Mathematik, Univ. Linz)
– Jean-René Chazottes (Dynamische Systeme, École Polytechnique Palaiseau)
– Clemens Fuchs (Univ. Salzburg) gemeinsam mit Lajos Hajdu (Univ. Debrecen): Zahlentheorie
– László Erdős (Stochastik und Mathematische Physik, IST Austria).

Bisher wurden drei Minisymposien eingereicht. *Weitere Tagungen:* Die CSASC 2016 findet in Smokovec in der Hohen Tatra im September 2016 statt, Roman Nedela wird die Organisation übernehmen. Das *Meeting of Presidents* europäischer mathematischer Gesellschaften findet vom 28.–29.3.2015 in Innsbruck statt. Der *ÖMG-DMV-Kongress 2017* findet vom 11.–15.9.2017 in Salzburg statt. Das Organisationskomitee hat sich konstituiert, und Clemens Fuchs hat den Vorsitz übernommen.

*Nichtständige Kommissionen*, die dem Vorstand berichtspflichtig sind, sollen wie folgt eingerichtet werden bzw. wurden schon eingerichtet: Die *Verbindungsgruppe Qualitätssicherungsrat* (Hans Humenberger, Gerald Teschl, Reinhard Winkler) wurde schon etabliert. Ihre Aufgaben sind die Begutachtung von Lehramts-Curricula nach Anforderung durch den Qualitätssicherungsrat (bisher für Innsbruck, Salzburg und Wien und seit Dezember 2014 auch für den Entwicklungsverbund Süd-Ost) und die Begutachtung von Lehrplänen.

Die *Verbindungsgruppe Schulfragen* hat die folgenden Aufgaben: Kontrolle der Beispiele für die Zentralmatura und Stellungnahme zu Grundkompetenzen und PISA-„Testitems“, mit Ausblick auf spätere Mitwirkung bei Bildungsstandards und Lehrplänen. Die schon etablierte Kerngruppe besteht aus Hans Humenberger, Barbara Kaltenbacher und Johannes Wallner. Aus einem Pool aus Kolleginnen und Kollegen aus der Mathematik (Clemens Heuberger, Bernhard Krön, Bernhard Lamel, Gunther Leobacher, Franz Pauer, Reinhard Winkler, Barbara Kaltenbacher und Johannes Wallner, Ersatz Michael Oberguggenberger) sowie aus der Fachdidaktik (Hans Humenberger, Günther Malle, Maria Koth, Franz Embacher, Ersatz Evelyn Stepancik) sollen mehrere Dreiergruppen eingerichtet werden, die die Aufgaben der AHS- und BHS-Matura jeweils zum Haupt- und Nebentermin begutachten.

*Schüler- und Schülerinnenpreise:* Die Ausschreibung wird in Bälde erfolgen; die Jury besteht aus Gerd Kadunz (Klagenfurt), Peter Schüller (MR i.R.) und Gabriela Schranz-Kirlinger (Wien).

*Beauftragter für Entwicklungszusammenarbeit:* Mit Winfried Müller (Klagenfurt) hat die ÖMG wieder einen Beauftragten für Entwicklungszusammenarbeit.

*Mathe-Brief:* Derzeit gibt es laut Gilbert Helmberg 248 Abonnentinnen und Abonnenten. Die Themen der Briefe seit April 2014 waren eine Pyramidenschnittaufgabe (Nr. 46, R. Resel, G. Helmberg), Kurzfassungen zu preisgekrönten Fachbereichsarbeiten (Nr. 47), Kuriosa (Nr. 48, G. Pilz), eine Webseite zur Wahrscheinlichkeitsrechnung (Nr. 49, M. Kammerhuber, B. Krön), Schüler- und Schülerinnenpreis der ÖMG (Nr. 50) und „Ein weiteres Smarties-Spiel" (Nr. 51, G. Kirchner).

*International Regions Mathematics League*-Wettbewerb in Raach am 31.5.2014. Die Teilnahme an diesem Wettbewerb wurde durch die ÖMG finanziert. G. Kirchner berichtet, dass Österreich den 2. Platz unter allen Teams erreicht hat, die an diesem Wettbewerb von zu Hause aus teilgenommen haben. Besonders hervorzuheben ist das österreichische Abschneiden in der "power round"; dabei wird der gesamte Lösungsweg (großteils Beweise, auf Englisch zu verfassen) in den USA zentral ausgewertet. Mit 39 von 50 Punkten wurde das zweitbeste Ergebnis aller ausländischen Teams (onsite und offsite) erreicht. Es dürften bei unseren Schülern und Schülerinnen auch die Englischkenntnisse vergleichsweise gut sein.

Die *Studierendenkonferenz* der Deutschen Mathematikervereinigung hat 2014 in Bochum stattgefunden. Unter den 16 Teilnehmern und Teilnehmerinnen gab es einen Österreicher; er erhielt einen Buchgutschein. Zukünftig wird sich Georg Hein (Univ. Duisburg-Essen) seitens der DMV um die Studierendenkonferenz kümmern.

*Nationalkomitee Mathematik:* Am 27.6.2014 hat die Österreichische Akademie der Wissenschaften die Zusammensetzung des Nationalkomitees Mathematik beschlossen: Monika Ludwig, Herbert Edelsbrunner, Michael Oberguggenberger (Anmerkung: Die Beschlussfassung der mathematisch-naturwissenschaftlichen Klasse erfolgte am 12.12.2014.)

*Finanzen:* Alexander Ostermann berichtet über die Finanzen, zunächst über die Unterfinanzierung der an alle inländischen Mitglieder regelmäßig versandten Mitteilungen der Deutschen Mathematikervereinigung. Im Vorstand der ÖMG wurde nach längerer Diskussion eine Verdoppelung des Beitrags auf €6.000 an die DMV beschlossen.

*Planung von Veranstaltungen:* Der Vorstand der ÖMG plant, eigene Veranstaltungen aus dem Budget zu unterstützen, wie etwa Sommerveranstaltungen für Studierende und für Frauenförderung, und nicht zuletzt die 2016 stattfindende Mathematikolympiade. Um weitere Anregungen wird gebeten.

**3. Bericht der Rechnungsprüfer und gegebenenfalls Entlastung des Vorstandes**. Peter Szmolyan berichtet, dass er und Hans Georg Feichtinger am 25.6.2014 gemeisam mit Frau E. Hofmann die Unterlagen genau geprüft haben und es keine Beanstandungen gab. Der Antrag zur Entlastung des Vorstands wird einstimmig angenommen.

**4. Berichte aus den Landessektionen und den Kommissionen**. Christian Krattenthaler (Wien) berichtet über eine französisch-österreichische Veranstaltung, weiters hat die Ausstellung „Mathematik zum Anfassen“ (mit Unterstützung der ÖMG) Station an der Uni Wien gemacht . Michael Eichmair von der ETH Zürich hat den Ruf auf die Professur *Globale Analysis und Differentialgeometrie* angenommen.

*Tirol:* Gerhard Kirchner berichtet, dass in Innsbruck eine Stochastik-Professur ausgeschrieben wird.

*Graz/Leoben:* Wolfgang Woess berichtet über eine Veranstaltung mit Christian Krattenthaler mit dem Titel „Musik und Mathematik“ mit der ÖMG als Hauptsponsor. Sonstige Aktivitäten: Die Nachfolge I. Berkes in *Stochastik und Versicherungsmathematik* ist am Laufen; es ist weiters gelungen, eine Professur in *Computational Topology and Geometry* vorzeitig auszuschreiben. Das vom FWF finanzierte Doktorandenkolleg „Discrete Mathematics“ befindet sich in der Verlängerungsphase.

*Klagenfurt:* Christian Pötzsche berichtet, dass ein Dreiervorschlag für eine Professur der Didaktik der Mathematik in Klagenfurt gerade erstellt wurde.

*Linz:* Friedrich Pillichshammer berichtet über die „Projektwoche Angewandte Mathematik“ im Februar 2014. Zudem ist der vom FWF finanzierte Spezialforschungsbereich *Quasi-Monte Carlo-Methoden: Theorie und Anwendungen* voll angelaufen. Seit September ist die Professur für *Analysis* (Nachfolge J. Cooper) mit Aike Hinrichs aus Rostock besetzt. Die Nachfolge G. Pilz (Algebra) befindet sich in der Verhandlungsphase. Im Februar 2015 wird zudem eine Professur für mathematische Methoden in Medizin und Biowissenschaften gemeinsam mit der neu gegründeten medizinischen Fakultät ausgeschrieben.

*Salzburg:* Peter Hellekalek berichtet, dass mit ÖMG-Mitteln Vorträge mitfinanziert wurden. Der Ausbau/Umbau des Fachbereichs auf Professorenebene sei nun abgeschlossen. Der ÖMG-DMV-Kongress 2017 wird in Salzburg stattfinden.

**5. Wahl des Beirats und der Vorsitzenden der Landessektionen.** Der Wahlvorschlag für die Vorsitzende der Landessektionen 2015–2016 lautet:

- Graz (Steiermark): Wolfgang Woess, TU Graz
- Innsbruck (Tirol, Vorarlberg): Hans Peter Schröcker (Univ. Innsbruck)
- Klagenfurt (Kärnten): Christian Pötzsche (Univ. Klagenfurt)
- Linz (Oberösterreich): Friedrich Pillichshammer (Univ. Linz)

– Salzburg: Peter Hellekalek (Univ. Salzburg)
– Wien (Burgenland, Niederösterreich, Wien): Christian Krattenthaler (Univ. Wien)

Der Wahlvorschlag wird einstimmig angenommen. Die Landesvorsitzenden nehmen die Wahl an.

*Didaktikkommission:* Es sind Anita Dorfmayr (auf eigenen Wunsch) und Peter Schüller (aufgrund seiner Pensionierung) aus der Didaktikkommission ausgeschieden. Als Nachfolger werden Christian Dorninger und Peter Hofbauer nominiert. Auch dieser Vorschlag wird einstimmig angenommen.

*Beirat:* Peter Gruber schlägt vor, jemanden vom IST Austria in den Beirat aufzunehmen. Es sollen Gespräche mit Herbert Edelsbrunner und László Erdős geführt werden. Vorbehaltlich ihrer Zustimmung wird der Antrag, einen der beiden in den Beirat aufzunehmen, einstimmig angenommen. Auch die Aufnahme von Michael Drmota und Harald Niederreiter (vorbehaltlich seiner Zustimmung) wird einstimmig beschlossen [Anmerkung: Sowohl H. Edelsbrunner als auch H. Niederreiter haben zugesagt.]

**6. Verleihung der Studienpreise und der Förderungspreise.** Mit den ÖMG-Studienpreisen werden diesmal die Dissertationen von Anna Geier und Annegret Burtscher ausgezeichnet. Leider konnten beide nicht persönlich anwesend sein. Der ÖMG-Förderungspreis wird an Christoph Haberl verliehen, die Laudatio wird von Franz Schuster gehalten.

**7. Allfälliges.** Es gibt keine Wortmeldungen.

*Vorsitzender:* Michael Oberguggenberger
*Schriftführerin:* Gabriela Schranz-Kirlinger

## Laudatio für Christoph Haberl aus Anlass der Verleihung des Förderungspreises 2014

Christoph Haberl wurde im Juli 1981 in Wien geboren und ist auch in Wien aufgewachsen und zur Schule gegangen. Er maturierte im Jahr 1999 am Bundesrealgymnasium in der Wenzgasse und trat danach seinen Wehrdienst an. Im Oktober des Jahres 2000 begann Christoph dann das Studium der Technischen Mathematik an der TU Wien, wo er von Anfang an zu den Besten seines Jahrgangs gehörte. Seine Diplomarbeit mit dem Titel „Über die Korrespondenz von Konvexgeometrie und Informationstheorie“ hat er unter Anleitung von Monika Ludwig verfasst und sein Studium im Juni 2005 mit Auszeichnung als Diplomingenieur abgeschlossen. Während seiner Dissertationszeit von Juli 2005 bis Juli 2007 war er in einem

FWF-Projekt von Monika Ludwig beschäftigt. Christoph und ich wurden so Kollegen an der TU Wien, wo wir uns etwa ein Jahr vorher kennengelernt hatten. Seit unserer gemeinsamen Zeit als Projektassistenten, während der wir auch einen gemeinsamen Forschungsaufenthalt an der Albert Ludwigs-Universität Freiburg verbrachten, verbindet uns eine enge Freundschaft und eine sehr fruchtbare mathematische Zusammenarbeit.

Schon in seiner Doktoratszeit lag ein Schwerpunkt von Christophs wissenschaftlicher Arbeit in der Theorie der Bewertungen oder additiven Abbildungen auf konvexen kompakten Mengen, also von Funktionen $\phi$ auf dem Raum $\mathcal{K}^n$ der konvexen Körper im $\mathbb{R}^n$ mit der Eigenschaft, dass

$$\phi(K \cup L) + \phi(K \cap L) = \phi(K) + \phi(L),$$

wann immer $K \cup L \in \mathcal{K}^n$. Seit der Lösung des Dritten Hilbertschen Problems durch Max Dehn im Jahr 1901 spielen Bewertungen eine zentrale Rolle in der diskreten und konvexen Geometrie sowie der Integralgeometrie. Beispiele von Bewertungen sind etwa das Volumen, die Oberfläche und die Euler Charakteristik, welche alle wichtige Messgrößen einer konvexen oder allgemeineren Menge liefern. Von besonderem Interesse in der Geometrie sind dabei Messgrößen, die auch gewisse Invarianzeigenschaften unter Transformationen des $\mathbb{R}^n$ haben. So ist auch ein Meilenstein der klassischen Bewertungstheorie eine Charakterisierung aller bewegungsinvarianten stetigen Bewertungen auf $\mathcal{K}^n$ von Hugo Hadwiger aus dem Jahr 1952. Solche Charakterisierungssätze nach dem Vorbild des Hadwigerschen Funktionalsatzes durchziehen auch das mathematische Werk von Christoph Haberl. Ich möchte gern eine seiner letzten Arbeiten zu diesem Thema, die Christoph gemeinsam mit Lukas Parapatits geschrieben hat, genauer besprechen. Sie hat den Titel 'The centro-affine Hadwiger Theorem' und ist 2014 im *Journal of the American Mathematical Society*, einer der selektivsten mathematischen Zeitschriften, erschienen.

Im Gegensatz zum Hadwigerschen Charakterisierungssatz geht es in der Arbeit von Christoph und Lukas um Bewertungen, die invariant unter volumenerhaltenden linearen Abbildungen, also invariant unter der speziellen linearen Gruppe $\mathrm{SL}(n)$, sind. Christoph und Lukas zeigen einerseits, dass jede stetige und $\mathrm{SL}(n)$ invariante Bewertung auf dem Raum $\mathcal{K}_o^n$ der konvexen Körper, die den Ursprung im Inneren enthalten, eine Linearkombination des Volumens $V_n$, der Euler Charakteristik $V_0$ und des Volumens des Polarkörpers $K^*$ von $K$ ist. Andererseits konnten sie eine viel tiefliegendere Variante ihres ersten Charakterisierungssatzes beweisen, bei der die Bedingung der Stetigkeit zur Oberhalbstetigkeit abgeschwächt wird. Es ist seit Langem bekannt, dass es eine große Klasse von bloß oberhalb stetigen Bewertungen gibt, die eine wichtige Rolle in der affinen Geometrie spielen. Dazu gehören die klassischen Affin- und Centro-Affinoberflächen von Wilhelm Blaschke sowie die $L_p$ Affinoberflächen, die Erwin Lutwak in den 1990ern eingeführt hat. Erst vor Kurzem haben Monika Ludwig und Matthias Reitzner die sogenannten Orlicz-Affinoberflächen mithilfe der Bewertungstheorie entdeckt und eine erste Charakterisierung in einer Arbeit in den *Annals of Mathematics* publiziert. Christoph und Lukas Parapatits ist es erstmals gelungen, eine vollständige Klassifikation *aller* oberhalbstetigen $\mathrm{SL}(n)$-invarianten Bewertungen auf $\mathcal{K}_o^n$ anzugeben. Genauer haben sie gezeigt, dass ein Funktional $\phi : \mathcal{K}_o \to \mathbb{R}$ genau dann eine oberhalbstetige und $\mathrm{SL}(n)$ invariante Bewertung ist, wenn es $c_0, c_1, c_2 \in \mathbb{R}$ und eine Funktion $g \in \mathrm{Conc}(\mathbb{R}^+)$ gibt, sodass

$$\phi(K) = c_0 V_0(K) + c_1 V_n(K) + c_2 V_n(K^*) + \Omega_g(K)$$

für alle $K \in \mathcal{K}_o^n$. Dabei ist $\mathrm{Conc}(\mathbb{R}^+)$ eine Klasse konkaver Funktionen auf $\mathbb{R}^+$, die einer gewissen Abklingbedingung genügen müssen, und $\Omega_g$ die dadurch bestimmte Orlicz-Affinoberfläche.

Bereits in seiner Dissertation mit dem Titel *Valuations and the dual $L_p$ Brunn-Minkowski theory* konnte Christoph wichtige Ergebnisse zur Theorie der Bewertungen beitragen. Dabei ging es allerdings nicht um reellwertige additive Funktionen, sondern um solche, die sternförmige kompakte Mengen wieder auf ebensolche Sternkörper abbilden. Seine Resultate wurden in den *Advances in Mathematics*, dem *Indiana University Mathematics Journal* sowie in einer hochbeachteten gemeinsamen Arbeit mit Monika Ludwig in den *International Mathematics Research Notices* publiziert. Darüber hinaus wurde die Dissertation von Christoph 2008 mit dem Studienpreis der ÖMG ausgezeichnet.

Nach dem Abschluss seines Doktorats trat Christoph eine Stelle als Universitätsassistent an der TU Wien an, bevor er im Jänner 2009 für drei Semester in die USA an das Polytechnic Institute der New York University wechselte. Seine in dieser Zeit entstandenen weiteren Arbeiten zu körperwertigen Bewertungen in Bezug auf die Minkowski- und Blaschke-Additionen haben besondere Resonanz gefunden. So gelang Christoph etwa in der Arbeit *Minkowski valuations intertwining the special linear group*, die im *Journal of the European Mathematical Society* 2012 erschienen ist, eine vollständige Klassifizierung aller $\mathrm{SL}(n)$-verträglichen Minkowski-Bewertungen. Dadurch wurde speziell eine Charakterisierung von Monika Ludwig aus dem Jahr 2002 des wichtigen Projektionenkörper-Operators von Minkowski stark verallgemeinert.

Neben den körperwertigen Bewertungen hat sich Christoph in New York mit dem sogenannten Orlicz-Minkowski-Problem beschäftigt. Dies ist eine Erweiterung des klassischen Minkowski-Problems, also dem Problem der Klassifizierung der Oberflächenmaße konvexer Körper. In Zusammenarbeit mit Erwin Lutwak, Deane Yang und Gaoyong Zhang konnte Christoph das Orlicz-Minkowski-Problem für den Fall von ursprungssymmetrischen Körpern vollständig lösen. Die zugehörige Arbeit ist 2010 in den *Advances in Mathematics* erschienen und stellt einen der ersten Beiträge zu der noch sehr jungen Orlicz-Theorie konvexer Körper dar. Diese wurde von der Gruppe um Erwin Lutwak mit zwei Arbeiten im Jahr 2010 begründet und, wie die Autoren selbst betonen, einerseits durch die Arbeit von Monika Ludwig und Matthias Reitzner zu Orlicz-Affinoberflächen und andererseits durch Arbeiten von Christoph und mir zu asymmetrischen $L_p$-isoperimetrischen Problemen motiviert.

In unserer ersten gemeinsamen Arbeit, die im *Journal of Differential Geometry* 2009 erschienen ist, haben Christoph und ich die enge Verbindung zwischen körperwertigen Bewertungen und isoperimetrischen Ungleichungen ausgenutzt, um die $L_p$-Version von Lutwak et al. der Projektionenungleichung von Petty wesentlich zu verallgemeinern. Damit haben wir den Grundstein gelegt, um die $L_p$-Brunn-Minkowski-Theorie symmetrischer konvexer Körper weiterzuentwickeln, um auch asymmetrische Anteile der Körper besser zu erfassen. Aufbauend auf dieser ersten Arbeit mit Christoph, haben wir 2009 im *Journal of Functional Analysis* asymmetrische $L_p$ Sobolevungleichungen bewiesen, die ebenfalls Ungleichungen von Lutwak et al. verschärfen. In Kooperation mit Jie Xiao von der University of Newfoundland in Kanada haben wir schließlich noch ein asymmetrisches affines Polyá-Szegő-Prinzip gezeigt (*Mathematische Annalen* 2012), das unter anderem zur ersten affinen logarithmischen Sobolevungleichung geführt hat.

Im Jahr 2010 wurde die au"serordentlich hohe Qualität von Christophs mathematischer Arbeit durch einen Ruf auf eine zeitlich befristete Professur an die Universität Salzburg gewürdigt. Diese Stelle in Salzburg sollte allerdings vorerst Christophs letzte Station in seiner akademischen Laufbahn werden. Er hat nämlich im März 2012 zu SAP Österreich gewechselt. Zur Freude seiner Kollegen aus der Geometrie bleibt Christoph neben seiner Hauptbeschäftigung aber noch genug Zeit, um nicht nur seiner Leidenschaft, dem Klavierspiel, nachzukommen, sondern auch um noch weiter Mathematik auf hohem Niveau zu betreiben. So hat er sich vor etwa zwei Jahren an der TU Wien habilitiert, hat gerade eine weitere Arbeit mit Lukas Parapatits fertiggestellt und wurde dieses Jahr bereits mit dem Edmund und Rosa Hlawka-Preis der Österreichischen Akademie der Wissenschaften ausgezeichnet. Nach einer Nominierung durch Monika Ludwig folgt nun der hochverdiente Förderpreis der ÖMG, zu dem ich Christoph ganz herzlich gratulieren möchte. Ein Abenteuer ganz anderer Art kommt auch bald auf Christoph und seine Frau Emanuela zu, da sie demnächst die Geburt ihres ersten Kindes erwarten, wofür ich beiden alles Gute und viel Glück wünsche.

(Franz Schuster)

# Neue Mitglieder

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